Fiabilité et cycle de vie des composants mécaniques dégradés : essais de démonstration et analyse basée sur la fonction de Hasard

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N° d'ordre : D.U. 1983

EDSPIC : 460

Université BLAISE PASCAL – Clermont II

Ecole Doctorale

Sciences pour l'ingénieur de Clermont- Ferrand

Thèse

Présenté par

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MSC de Heriot-Watt université en Royaume Uni

Pour obtenir le grade de

Docteur d'université

Spécialité : Génie Mécanique

Fiabilité et cycle de vie des composants mécaniques dégradés : Essais de démonstration et analyse basée sur la fonction de Hasard

Soutenue publiquement le 03 décembre 2009 devant le jury :

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Remerciements

Je tiens à exprimer mes plus vifs remerciements à Monsieur Alaa CHATEAUNEUF qui fut pour moi un directeur de thèse attentif et disponible malgré ses nombreuses responsabilités. Je lui suis extrêmement reconnaissante pour la liberté qu'il a bien voulu me laisser. Sa compétence, sa clairvoyance, son humanisme tout au long de ces trois années de thèse m'ont beaucoup appris.

Je remercie Monsieur Fabrice GUERIN, Professeur à l'INSTIA – Université d'Angers, pour l'intérêt qu'il a manifesté pour mon travail, pour ses questions constructives et ses précieux conseils. Je le remercie d'avoir accepté de juger ce travail en tant que Rapporteur et Président de jury.

Je remercie Monsieur Mikhail NIKOULINE, Professeur à l'IMB – Université de Bordeaux II, pour l'attention qu'il a accordée à mon travail, pour le temps qu'il a bien voulu consacrer à ce mémoire, sa lecture attentive, entre autres, du manuscrit et enfin pour ses questions et remarques constructives et intéressantes. Je le remercie d'avoir accepté de juger ce travail en tant que Rapporteur.

Je remercie Madame Zohra CHERFI-BOULANGER, Professeur à l'Université de Technologie de Compiègne, pour l'intérêt qu'elle a porté à mon travail. Je lui suis reconnaissant d'avoir accepté d'être Examinatrice de ce travail.

Je remercie Monsieur André LANNOY, Directeur de recherche, Institut de Maîtrise des Risques, pour son analyse critique, son regard industriel et ses encouragements ainsi que pour ses recommandations, ses questions et ses remarques constructives. Je le remercie d'avoir accepté d'être Examinateur de ce travail.

Je remercie Monsieur Lambert PIERRAT, Directeur de recherche, Université de Grenoble, pour ses propositions très intéressantes ainsi que ses questions et remarques constructives. Je le remercie d'avoir accepté d'être examinateur de ce travail.

Je remercie tous les membres du LaMI pour leur assistance et leur soutien. Je remercie particulièrement les amis arabes et français pour leur aide et leur soutien surtout à la préparation de la soutenance. Je leur souhaite ainsi qu'à mes autres camarades doctorants tout le courage pour finir leur thèse.

Une mention et des remerciements très spéciaux sont réservés à ma chère épouse Salwa pour sa patience, sa confiance, son soutien et son amour. Je pense aussi à mes enfants Hyiam, Ali, Zo alfakar et Hamza. Je les remercie et je leur dédie ce travail.

Enfin, j'adresse toute mon affection à toute ma famille. Une mention très spéciale à mon père Abou Hussam et mon beau frère Ayman qui ont rempli l'espace que j'ai laissé dans ma maison et ma famille pendant mon séjour en France.

Hussam

RESUME

Le processus de dimensionnement des structures et des systèmes mécaniques comportent de plusieurs étapes, allant de la définition des conditions et des besoins tout au long du cycle de vie, en vue de la spécification de la capacité et de la résistance requises pour accomplir les missions escomptées. La fiabilité figure parmi les objectifs les plus importants pour les fabricants, en plus de l'aspect économique, facteur clé, qui influence largement le processus de conception. Dans ce contexte, la conception doit être élaborée afin de définir le meilleur compromis entre la fiabilité et le coût. Ce qui implique une étude précise et détaillée de tout le cycle de vie du produit, de la naissance jusqu'à la mise au rebus.

Cette étude couvre les différentes phases du cycle de vie du produit, en intégrant la nécessité de démontrer la fiabilité du produit avant de commencer la production en série, sous des contraintes de coût et de délais.

Ce travail vise à donner des éléments de réponse aux trois questions suivantes :

- Comment peut-on démontrer la fiabilité du produit à partir de quelques essais ? Parmi les quatre approches considérées, la méthode de composition des incertitudes montre sa robustesse pour démontrer la fiabilité du produit, sans pour autant conduire à un surdimensionnement excessif.

- Quel est le critère permettant une conception robuste sous des charges répétitives pour un système non dégradable ? Dans la phase utile du cycle de vie du produit, la défaillance est principalement due à la variabilité des charges appliquées lorsque la résistance n'est pas dégradée. Le modèle d'interférence contrainte-résistance considère la probabilité de défaillance comme cible de conception. Cependant, pour le cas des charges répétitives, ce critère est sensible au nombre d'applications de ces charges. Pour cela, la conception basée sur le hasard est proposée comme outil robuste pour la conception des composants intrinsèquement fiables.

- Quelle est l'approche générale permettant de traiter les mécanismes de dégradation ? Dans la phase de vieillissement, la modélisation de la dégradation est obligatoire pour plusieurs raisons, telles que la maîtrise des risques industriels et la gestion du cycle de vie. La fonction de hasard fournit un indicateur approprié pour la prévision de l'état de dégradation et par conséquent, l'estimation de la durée de vie résiduelle.

ABSTRACT

The process of designing and producing mechanical and structural systems consists of several stages, starting from defining the requirements and the demands throughout the life cycle, that must be supported to determine the capacity or resistance needed to fulfil the equipment mission. The reliability is the one of the most important goals that manufacturers seek, while the economical aspect is a key factor and it has a great deal influence on this process. Therefore, the best design has to be carried out, in order to achieve the paradox of reliable products with minimal costs. This implies careful and exact investigation along the product life-cycle, from birth to death.

This study encompasses the different phases of product life cycle, starting from the necessity to demonstrate the product reliability before starting the mass production under the constraints of economy and time.

This work aims to answer the following three questions:

- How can we demonstrate the product reliability on the basis of few tests? Among the four approaches considered in this study, the method of compound uncertainties shows its robustness to demonstrate the product reliability, without implying unnecessary over-design.

- What is the robust design criterion under repetitive load for time-independent resistance? In the useful phase of product life cycle, failure is assigned to load variability under the assumption of non-degraded resistance. Stress-resistance model considers failure probability as a design target to be achieved; however, for the case of repetitive loading, this criterion is sensitive to the number of load applications. The present works shows that hazard is almost constant and gives a robust design criterion.

- Is there a general approach that can cope with all degradation mechanisms? In the wear out phase, modelling degradation is mandatory for several reasons such as industrial risks control and life cycle management. The hazard function gives an appropriate indicator for the prediction of the degradation state and consequently, the estimation of residual product life.

Principal notation

$\Re$, $\Re(t)$, $\Re(n)$	Reliability of component, expressed in terms of the time t or the number of loads n.
P_f	Probability of failure.
CDF	Cumulative distribution function for a random variable.
PDF	Probability density function for a random variable.
COV	Coefficient of variation.
$F(.)$	Cumulative distribution function of life.
$f(.)$	Probability density function of life.
$h(t)$, $h(n)$	Hazard expressed in terms of t or n.
$H(t)$	Cumulative hazard function.
T	Time to failure.
t	Time.
P, Pr	Probability operator.
MTTF	Mean time to failure.
S	Component stress.
R	Component resistance.
$f_s(x)$	Probability density function of stress.
$f_R(x)$	Probability density function of resistance.
$F_R(x)$	Cumulative distribution function of resistance.
$G_f(\{x\})$	Surface of failure.
$G(\{x\})$	Function of failure.
Ω_s, Ω_f	Safety and failure domains.
β_C	Cornell reliability index.
β_{HL}	Hasofer and Lind reliability index.
β_H	Reliability index corresponding to hazard target.
β_W	Weibull shape parameter.
m, σ, ρ	Mean value, standard deviation and coefficient of correlation.
LR	Load roughness.
α_R, α_S	Direction cosines of resistance and load, respectively.
c_R, c_S	Coefficients of variation of resistance and stress, respectively.
$C\%$	Confidence level.
$\chi^2_{\alpha,n}$	$100(1-\alpha)^{\text{th}}$ percentile for chi-square with n degrees of freedom.
H_0	Null hypothesis.
H_1	Alternative hypothesis.
$L(.)$	The likelihood function.
K	Safety factor.
$\overline{R}$	Average resistance estimate, obtained from tests.
$\bar{r}$	Realization of mean resistance, obtained from observation.
F_t	Test factor.

S_{test}	Stress of test.
η	Characteristic life.
LCC	Life cycle cost.
$\phi(\cdot)$, $\Phi(\cdot)$	Standard normal probability density and cumulative distribution functions, respectively.
Γ	Gamma function.
SM	Normalized Safety margin, or reliability index.
E	Damage resistance threshold.
d	Damage.
ξ	Degradation function, or *s*-*N* function.

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Résumé étendu

Ce résumé présente la synthèse des travaux,

qui sont ensuite détaillés dans les chapitres en version anglaise.

Résumé étendu

I.1. Introduction

Ce travail de recherche a pour objectif d'examiner la méthodologie de conception fiable des structures et composants mécaniques. Dans ce cadre, la théorie de la fiabilité des structures, basée sur la méthode contrainte-résistance, est appliquée dans les différentes phases du cycle de vie du produit. Cela inclut les essais de démonstration de la fiabilité, la conception basée sur la fiabilité pendant la durée de vie utile et la gestion de la phase de dégradation en tenant compte des incertitudes. Dans cette démarche, la fonction de hasard est utilisée comme support pour l'évaluation de la fiabilité sous l'action des charges répétitives.

Dans ce résumé, nous décrivons les développements principaux pour l'analyse fiabiliste en termes de démonstration, de conception et de modélisation de la dégradation. Les détails de ces développements se trouvent dans les chapitres correspondant en langue anglaise.

I.2. Fiabilité des structures

L'état des structures et des systèmes mécaniques dépend des sollicitations extérieures appliquées, des propriétés matérielles, des modèles de conception et des facteurs organisationnels et humains, intervenant dans la conception, la réalisation et l'exploitation du produit. La fiabilité d'une structure, ou d'un composant, se traduit par sa capacité d'accomplir ses objectifs de conception pendant un temps de référence spécifié, dans des conditions données. La probabilité de défaillance est donc l'événement complémentaire.

$$P_f = 1 - \Re \tag{1}$$

où $\Re$ est la fiabilité et P_f est la probabilité de défaillance au cours de la période de référence.

I.2.1. Cycle de vie et fiabilité

La probabilité de défaillance cumulée sur un intervalle de temps correspond à la fonction de répartition de la durée de vie $F(.)$, dont la densité est notée $f(.)$. La probabilité de défaillance par unité de temps, conditionnée par la survie du système jusqu'à l'instant d'observation, est appelée « fonction de hasard », notée $h(.)$ (figure 1). Pour les systèmes mécaniques, il est pratiquement impossible de déterminer la fiabilité réelle à partir des observations de la population. Pour cette raison, la théorie de la fiabilité des structures s'est développée dans le but de donner des estimations convenables de l'état des systèmes en service.

$$h(t) = \frac{f(t)}{\Re(t)} \tag{2}$$

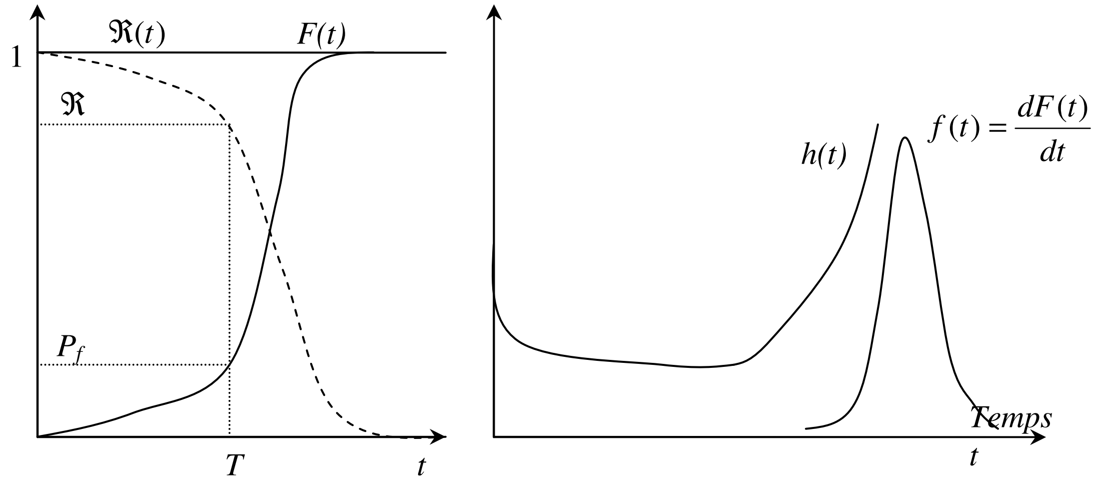


Figure 1. Fonctions d'état : fiabilité, probabilité de défaillance et hasard.

I.2.2. Modèle Contrainte- Résistance

La situation la plus simple correspond au cas où la fiabilité d'un composant est déterminée par deux variables aléatoires et indépendantes : sollicitation S et résistance R. La défaillance a lieu lorsque la sollicitation dépasse la résistance.

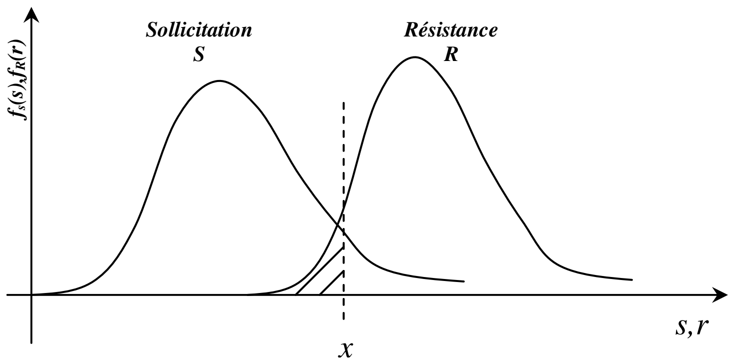


Figure 2. Distributions de la sollicitation et de la résistance

La probabilité de défaillance P_f est donnée par la probabilité d'atteindre un certain niveau de charge lorsque le système présente une résistance inférieure à ce niveau. Pour le cas des variables indépendantes, cette probabilité est exprimée par :

$$P_f = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx \tag{3}$$

Bien que l'expression est relativement simple, son évaluation est extrêmement coûteuse en temps et exige une précision très élevée, puisque la probabilité calculée est très faible. Pour cette raison, les méthodes de fiabilité du premier et du second ordre FORM/SORM ont été développées en tant qu'alternatives efficaces et pratiques [Mad-99]. Elles sont basées sur le calcul d'une certaine mesure de la fiabilité, connue sous le nom d'indice de fiabilité [Mad-86, Rac-01], et évalué en résolvant le problème d'optimisation suivant:

$$\begin{cases} \underset{u_i}{\text{minimiser}} \ \sum_{\text{i}} u_i^2 \\ \text{sous}: \ G(\mathbf{d}, \mathbf{x}) = 0 \end{cases} \tag{4}$$

où $G(.)$ est la fonction d'état limite (appelée également, marge de sûreté ou fonction de performance), **d** est le vecteur des paramètres de conception déterministes, **x** est le vecteur de réalisations des variables aléatoires X_i, et u_i sont « les variables normalisées » obtenues par la transformation probabiliste : $u_i = T_i(x_j)$ [Mad-86], avec x_j les réalisation des « variables physiques », comme l'illustre la figure 3.

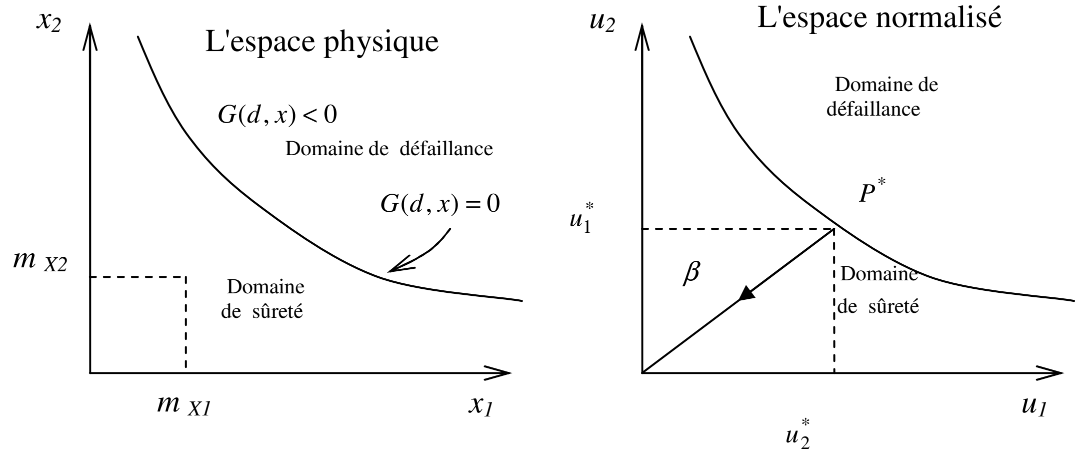


Figure 3. Problème d'optimisation de fiabilité

La résolution du problème (4) conduit aux coordonnées du « point de conception » **u***, appelé également « point de défaillance le plus probable », et à « l'indice de fiabilité » correspondant à la distance entre l'origine du repère normé et ce point de conception.

Dans le cadre de la méthode contrainte-résistance, la fonction d'état limite s'écrit : $G(R,S) = R - S$. Lorsque les deux variables R et S sont normales et indépendantes, cette expression s'écrit dans l'espace normé par :

$$H(U_R, U_S) = \sigma_R U_R - \sigma_S U_S + m_R - m_S \tag{5}$$

où m_R, m_S, σ_R et σ_S sont respectivement les moyennes et les écarts-types de la résistance et de la contrainte. Pour les lois normales, la résolution du problème de fiabilité (4) donne l'indice de fiabilité : $\beta = \frac{m_R - m_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$. Il est facile de démontrer que la marge de sûreté (5) est inchangée lorsqu'elle divisée par une constance, en l'occurrence $\sqrt{\sigma_R^2 + \sigma_S^2}$. Au point de conception (u_R^*, u_S^*), nous avons $H(u_R^*, u_S^*) = 0$ et l'équation 5 s'écrit :

$$\beta + \alpha_R \, u_R^* + \alpha_S \, u_S^* = 0 \tag{6}$$

avec : $\beta = \dfrac{m_{R_1} - m_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$; $\alpha_R = \dfrac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_S^2}}$ *et* $\alpha_S = -\dfrac{\sigma_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$

où α_R et α_S sont respectivement les cosinus directeurs de la résistance et de la sollicitation ; ils représentent l'influence des variables correspondantes sur la fiabilité du composant. Au niveau de l'approximation du premier ordre, la probabilité de défaillance est définie en terme de l'indice de fiabilité β par :

$$P_f = \Phi(-\beta) \tag{7}$$

où $\Phi(.)$ est la fonction de répartition de Gauss. Ainsi, la fiabilité est calculée par :

$$\Re = 1 - \Phi(-\beta) = \Phi(\beta) \tag{8}$$

I.2.3. Rugosité du chargement

Carter [Car-86] a introduit la notion de « la rugosité du chargement », donnée par un paramètre adimensionnel représentant le rapport entre l'écart-type de la sollicitation et celui de la marge de sûreté. Dans le cas de la méthode contrainte-résistance avec des variables normales indépendantes, la rugosité du chargement correspond, au signe près, au cosinus directeur de la sollicitation [Car-97].

$$\text{Rugosité de sollicitation} = -\alpha_s = \frac{\sigma_S}{\sqrt{\sigma_S^2 + \sigma_R^2}} \tag{9}$$

La rugosité du chargement varie de 0 à 1 : 0 représente le cas de la sollicitation déterministe et 1 représente le cas de la résistance déterministe. Selon les valeurs de la rugosité du chargement, différents cas sont distingués :

- **Charge rugueux**, correspondant à une grande dispersion de la sollicitation.
- **Charge de rugosité moyenne**, où la dispersion de la sollicitation est modérée.
- **Charge lisse**, où la sollicitation présente une faible dispersion.

La notion de rugosité reflète l'influence de la dispersion de la charge sur le niveau de fiabilité (i.e. sensibilité de la fiabilité par rapport à la charge).

II. Essais de démonstration de la fiabilité

Dans un contexte industriel très compétitif, les pressions exercées sur les fabricants conduisent à des produits de plus en plus fiables et de moins en moins coûteux, dans des délais de plus en plus réduits. Ces défis ont incité les fabricants à développer et à déployer des programmes de fiabilité efficaces. En fait, un programme efficace se compose d'une série de tâches de fiabilité, mises en application dans tout le cycle de vie du produit, y compris la planification, la conception et le développement, la vérification et la validation, la production, l'exploitation, et le recyclage ou la destruction (figure 4) [Yan-07].

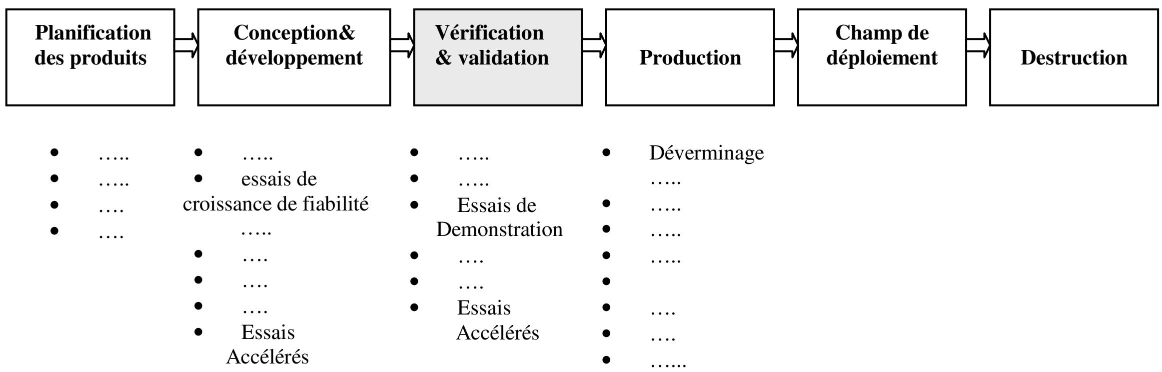


Figure 4. Tâches de fiabilité pendant le cycle de vie du produit [Yan-07]

Les activités de fiabilité ne sont pas des tâches indépendantes, mais elles doivent être intégrées dans chacune des étapes du cycle de vie. Ces tâches incluent les différents types d'essai de fiabilité qui sont considérés comme la pierre angulaire du programme de fiabilité. Généralement, le but de l'essai de fiabilité est de déceler des problèmes potentiels avec la conception dès que possible et, finalement, de fournir la garantie que le système réponde aux exigences de fiabilité. En plus, il constitue la forme la plus détaillée des données de fiabilité. Le type d'essai de fiabilité qu'un produit subit est différent, selon l'instant considéré de son cycle de vie, le but étant de s'assurer que les données produites par les essais puissent caractériser la fiabilité du produit à différents instants de son cycle de vie. Ces essais peuvent être réalisés à divers niveaux. Par exemple, les systèmes mécaniques peuvent être examinés au niveau des matériaux, des composants, des unités, des assemblages et du système complet.

Toutefois, la démonstration de la fiabilité par les essais est problématique pour plusieurs raisons. Alors qu'un essai simple est insuffisant pour produire des données statistiques utiles, les essais multiples ou les essais de longue durée sont très chers, certains essais sont même impraticables ou impossibles. Notre objectif est donc de fournir une méthodologie permettant de prendre en compte le faible nombre d'essais dans le but de la démonstration de la fiabilité des produits.

II.1 Essai de validation ou Essai de démonstration

Selon le paragraphe 7.3.5 de l'ISO 9001:2000, la validation est le terme utilisé pour des activités continues d'essai permettant de démontrer l'accomplissement des objectifs de la conception. Dans ce travail, nous avons étudié les différentes approches permettant de définir les essais de validation, qui sont récapitulées sur la figure 5.
En fonction de la taille de l'échantillon, nous avons classé les essais de validation en deux catégories : les approches où le nombre d'essais est pré-déterminé, tel que l'essai Bogey, et les approches où ce nombre est inconnu à l'avance, telles que les essais séquentiels et les approches basées sur le modèle contrainte-résistance ; notre travail est concentrée sur ces approches.

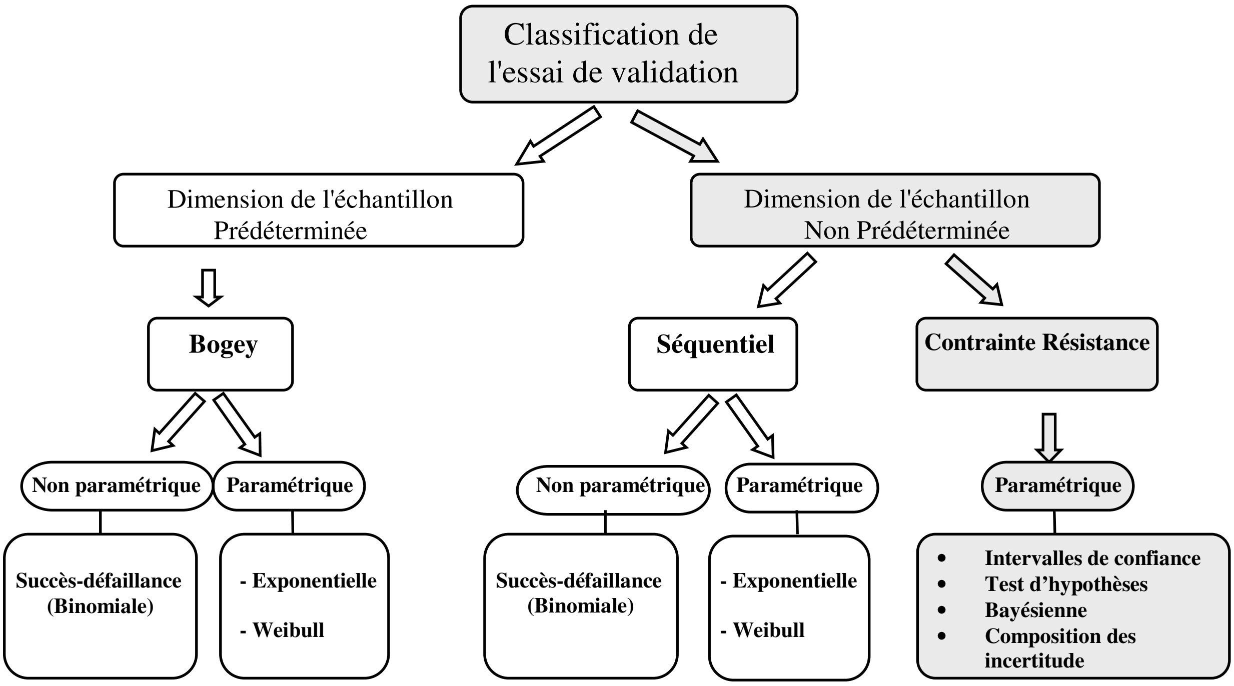


Figure 5. Classification des essais de validation.

Quatre approches probabilistes basées sur la fiabilité des structures sont considérées pour réaliser l'objectif de démonstration de la fiabilité, à savoir : la méthode des intervalles de confiance, les tests d'hypothèse, l'approche Bayésienne et la composition des incertitudes. Cette dernière est proposée pour la démonstration de la fiabilité, où l'évaluation robuste est démontrée pour les échantillons de très petite taille. Dans notre étude, il est supposé que les types de distribution et le coefficient de variation sont déterminés, à partir de l'état actuel de connaissance. La fiabilité cible est prédéfinie, ce qui nous permet de fixer l'objectif de résistance moyenne pour la population en service m_R^{obj} qui doit être suffisamment éloignée de la charge moyenne m_S; i.e. $m_R^{obj} = k\, m_S$, où k est le coefficient de sécurité. L'approche de la composition des incertitudes tient compte des incertitudes épistémiques dans les essais lors de l'évaluation de la fiabilité du produit en service. Finalement, l'optimisation du coût du cycle de vie est effectuée sur la base de la méthode de composition des incertitudes, afin de déterminer le nombre optimal d'essais, permettant de satisfaire les objectifs de fiabilité et de validation.

II.2 Exemple numérique

Dans cet exemple l'état limite considéré est : $G(R,S) = R - S$, ou R et S sont des variables normales avec les paramètres suivants : moyenne de la sollicitation $m_S = 180\,MPa$, son coefficient de variation $c_S = 0.15$ et coefficient de variation de la résistance $c_R = 0.10$. La résistance doit être déterminée pour atteindre l'indice de fiabilité $\beta_T = 4$ (correspondant à la probabilité de défiance $P_f = 3.2 \times 10^{-5}$).

L'objectif de résistance moyenne est obtenue par la relation :

$$\beta_T = \frac{m_R - m_S}{\sqrt{(c_R m_R)^2 + (c_S m_S)^2}} = \frac{m_R - 180}{\sqrt{(0.10\, m_R)^2 + (0.15 \times 180)^2}} = 4$$

dont la solution est $m_R^{obj} = 360$, ce qui représente la cible de conception pour les produits en service. Pour démontrer la fiabilité, cinq essais sont réalisés séquentiellement pour mesurer la résistance de produit. Quatre approches probabilistes sont appliquées pour vérifier la fiabilité cible, ces approches sont : l'intervalle de confiance, le test d'hypothèse, la méthode Bayésiennne et la composition des incertitudes. Les résultats sont présentés sur la figure 6.

- Dans l'approche de l'intervalle de confiance, la fiabilité est démontrée si la borne inférieure de la résistance moyenne obtenue à partir des essais est plus grande que l'objectif visé m_R^{obj}. Dans cet exemple, la fiabilité n'a pas pu être démontrée lors des essais 2 et 5 (figure 6).

- Dans l'approche de test d'hypothèses, la résistance réelle obtenue à partir des essais doit être plus grande que la charge d'essai imposée par cette approche. A part le cinquième essais, tous les autres on échoué à démontrer la fiabilité (Figure 6).

- Dans l'approche Bayésienne, pour démontrer la fiabilité la résistance prévue doit être plus grande que la résistance objectif (Figure 6). Dans notre cas, tous les essais n'ont pas démontré la fiabilité.

- Dans l'approche de composition des incertitudes, la fiabilité est démontrée si la résistance moyenne obtenue à partir des essais est plus grande que la sollicitation imposée par cette approche (Figure 6). Tous les essais ont pu démontrer la fiabilité requise.

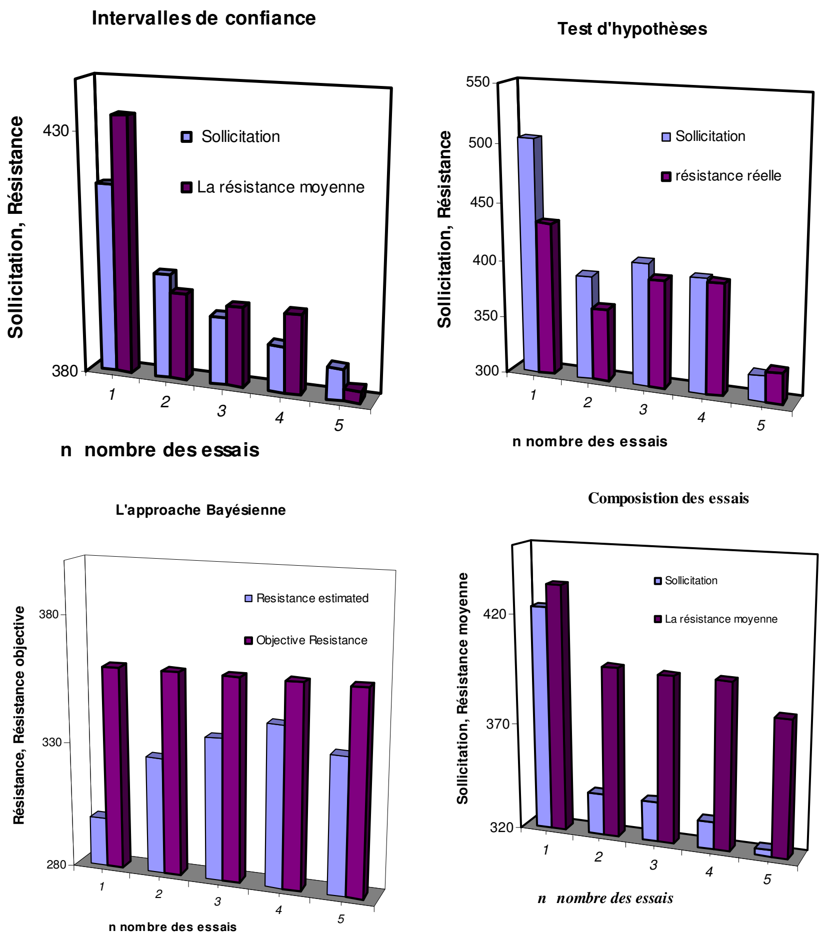


Figure 6. Approches de démonstration de la fiabilité.

Cette application montre que l'approche de la composition des incertitudes est capable de démontrer la fiabilité, en prenant en considération toutes les incertitudes liées à la résistance, ainsi que celles liées à la sollicitation (i.e. produit en service). Elle offre ainsi un cadre cohérent pour la démonstration de la fiabilité

II.2 Optimisation du coût de conception et de validation

Dans ce travail, nous combinons le coût de conception et des essais de validation afin de définir le coût minimum avec le nombre optimal d'essais. La solution permet de démontrer les objectifs de fiabilité avec le coût et le nombre d'essais optimaux. Le problème d'optimisation s'écrit :

$$\min C_{tot} = C_{design} + C_{test} \tag{10}$$

$$\text{sous} \quad \frac{m_{R_{obj}} - m_s}{\sqrt{(c_R m_{R_{obj}})^2 + \sigma_s^2}} \geq \beta_t$$

$$\frac{m_{\bar{R}} - m_{R_{obj}}}{\dfrac{c_R m_{\bar{R}}}{\sqrt{n_{test}}}} \geq \beta_\alpha$$

avec β_t la fiabilité cible et β_α le niveau de confiance dans les essais.

III. Conception basée sur la fonction de hasard pour les systèmes soumis aux charges répétitives

L'application classique de la méthode contrainte-résistance porte sur la probabilité instantanée, considérant une seule occurrence de la confrontation des deux variables. Or dans la réalité, la même structure subit l'application des charges extérieures à plusieurs reprises. La notion du temps d'exposition ou du nombre d'applications est donc fondamentale pour bien évaluer la fiabilité du produit. Une solution possible consiste à considérer les lois des valeurs extrêmes en fonction de la fenêtre de temps d'observation. Or cette hypothèse est souvent pessimiste et ne tient pas compte du fait que le nombre de répétitions des charges est le plus souvent inconnu. Etant donné que la probabilité de défaillance doit intégrer le temps de fonctionnement et que le nombre d'applications est inaccessible, la conception basée sur le taux de défaillance fournit une meilleure garantie du niveau de fiabilité, ce qui n'est pas le cas de la probabilité de défaillance dont la valeur est fortement dépendante de la durée d'exposition.

Pour situer le contexte de cette partie du travail, le problème de contrainte-résistance est illustré sur la figure 7, en fonction de la nature du chargement, des causes et des mécanismes de défaillance. Nous pouvons distinguer les charges simples et répétitives, les résistances dégradées et non dégradées, les défaillances engendrées soit par la variabilité des charges, soit par la dégradation de la résistance. D'un autre côté, les mécanismes de défaillance correspondent soit à la rupture brusque, soit au vieillissement (dégradation lente et irréversible).

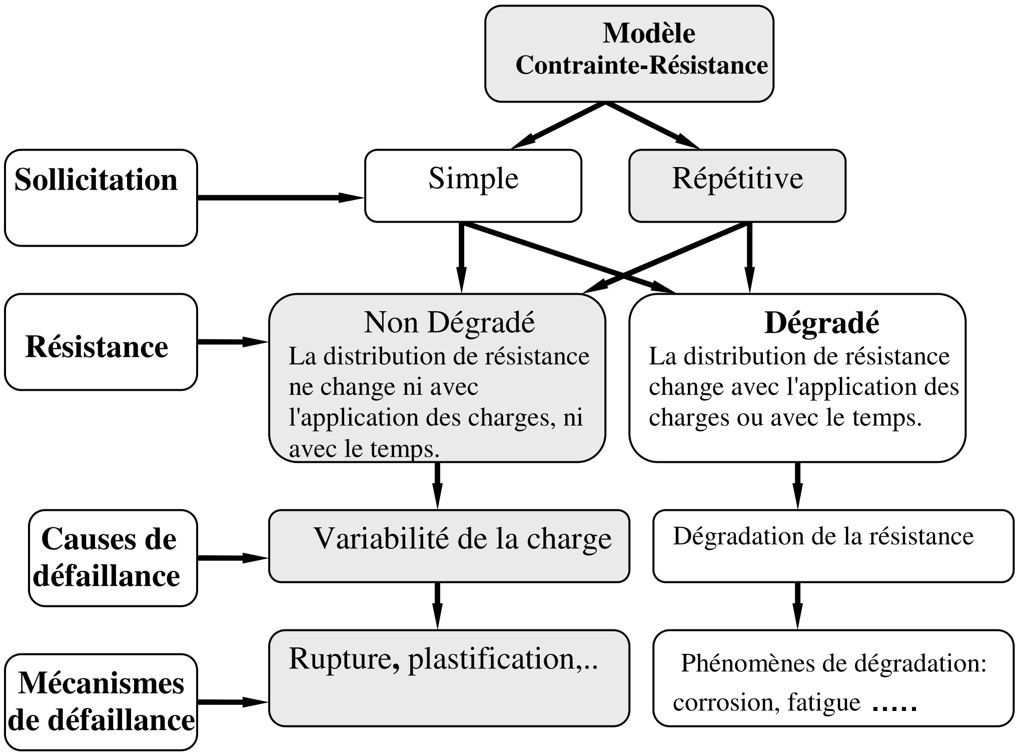


Figure 7. Classification du modèle contrainte-résistance.

Dans la phase où le taux de défaillance est constant, la défaillance est principalement due à la variabilité du chargement [Lew-94,Car-97]. Par conséquent, il nous paraît plus judicieux d'effectuer la conception en utilisant la méthode contrainte-résistance en termes de fonction de hasard cible, plutôt que de probabilité de défaillance cible, surtout lorsque le chargement est répétitif, afin d'assurer une solution robuste quel que soit le nombre d'applications des charges (Figure 7).

III.1 Conception basée sur la fonction du hasard

La méthodologie de conception probabiliste, basée sur le modèle d'interférence contrainte-résistance, considère la probabilité de défaillance comme objectif de conception à atteindre pour une application singulière du chargement. En fait, la cible « probabilité de défaillance » varie avec le nombre d'applications de la charge. Le cas des charges répétitives est traité pour un grand nombre par l'approche pessimiste des lois des valeurs extrêmes. Dans le présent travail, nous considérons le risque comme cible de conception au lieu de la probabilité de défaillance.

Dans l'espace normé, nous pouvons exprimer la probabilité de défaillance en termes de deux paramètres : indice de fiabilité β et rugosité de la sollicitation $-\alpha_s$:

$$F(n) = 1 - \Re(n) = 1 - \int_{-\infty}^{\infty} \phi(u) \left[\Phi\left(\frac{u - \frac{-\beta}{\sqrt{1-\alpha_s^2}}}{\frac{-\alpha_s}{\sqrt{1-\alpha_s^2}}} \right) \right]^n du \tag{11}$$

où n est le nombre d'applications de la charge, $F(.)$ est la fonction de répartition de la durée de vie, correspondant à la probabilité de défaillance cumulée, et $\phi(.)$ et $\Phi(.)$ sont respectivement la fonction de densité et la fonction de répartition de la loi normale standard. La fonction de hasard s'exprime par :

$$h(n) = \frac{F(n) - F(n-1)}{1 - F(n)} \quad pour \quad n \geq 2 \tag{12}$$

L'analyse de sensibilité (Figure 8) montre l'indépendance du hasard (risque) vis-à-vis du nombre d'applications du chargement. Les exemples traités confirment la robustesse de cette approche par rapport à l'approche traditionnelle en mécanique basée sur la probabilité de défaillance cible. Cet objectif permet également de définir le problème d'optimisation basée sur le hasard HBDO.

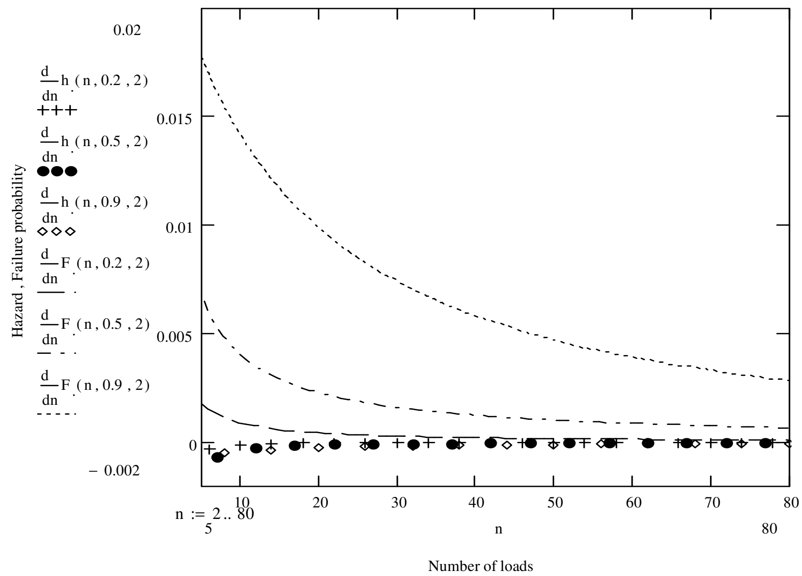


Figure 8. Sensibilité de probabilité de hasard

Les développements sont effectués pour le cas des lois normales et Weibull. Une procédure numérique permet l'extension aux différents types de distribution ; l'objectif étant de spécifier ***une conception intrinsèquement fiable***, comme l'illustre la figure 9.

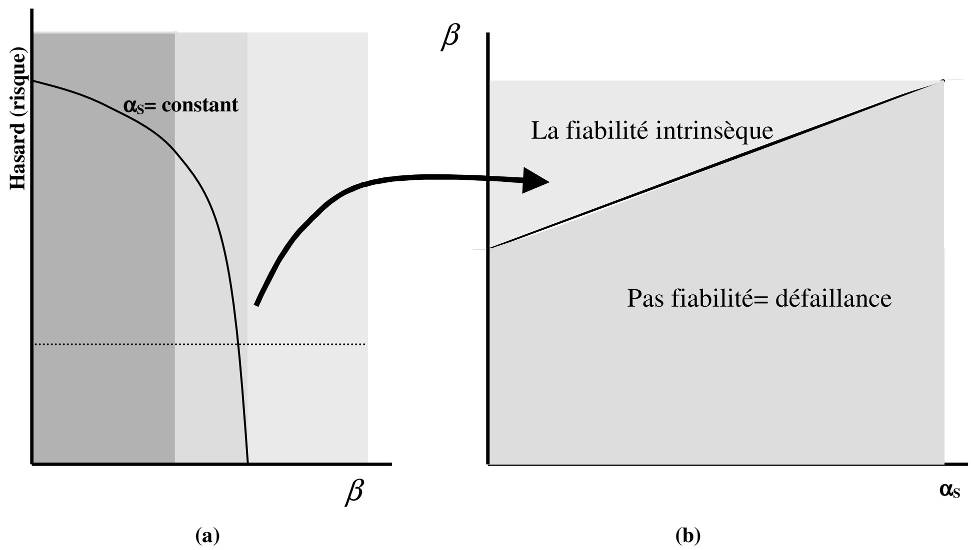


Figure 9. a) courbe type de la fonction de hasard et marge de sûreté associée. b) Marge de sûreté pour une conception intrinsèquement fiable, en fonction de la rugosité du chargement.

III.2 Exemple numérique

Considérons le système des deux barres, représenté sur la figure.10. Le système est appuyé aux deux noeuds A et B, et soumis à une charge verticale P au noeud C. Les sections transversales sont cylindriques minces, dont l'aire est $S_i = 2\pi . r_i . e_i$ (i=1,2) et le moment de l'inertie est : $I_i = \pi \, r_i^3 e_i$. Le coût unitaire de l'acier est donné par c_0 =l € /kg. Le module de Young est normalement distribué avec une moyenne $m_E = 210$ GPa et un écart type $\sigma_E = 11\,\text{GPa}$, la force appliquée est également normale de moyenne $m_P = 50000\,\text{N}$ et d'écart type $\sigma_P = 8500\,\text{N}$. Le critère de conception est lié au flambage des éléments.

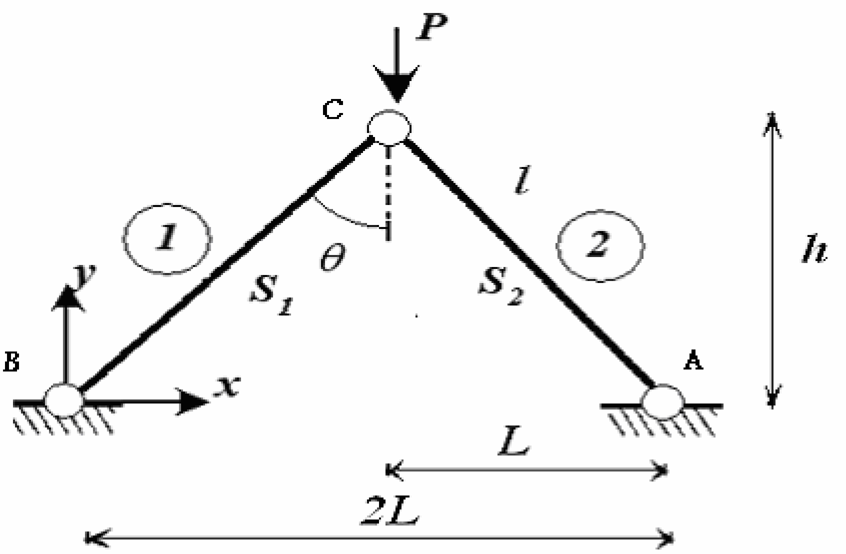


Figure 10. Structure à deux barres sous une force verticale

La fonction d'état de limite est :

$$G = P_{cr} - P \tag{13}$$

où P_{cr} est la force critique d'Euler, calculée par :

$$P_{cr} = \frac{\pi^2 E\, I}{l^2} = \frac{\pi^3 E r^3 \mathrm{e}}{\sqrt{L^2 + h^2}} = \frac{\pi^3 r^3 \mathrm{e}}{\sqrt{L^2 + h^2}} E \tag{14}$$

Substituant cette expression dans l'équation (13), la fonction d'état de limite prend la forme :

$$G = a\, E - b\, P \tag{15}$$

avec $a = \frac{\pi^3 r^3 \mathrm{e}}{\sqrt{L^2 + h^2}}$ et $b = \frac{1}{2} \frac{\sqrt{L^2 + h^2}}{h}$. L'indice de fiabilité correspondant est donné par :

$$\beta = \frac{a\, m_E - b m_P}{\sqrt{a^2 \sigma_E^2 + b \sigma_P^2}} \tag{16}$$

Dans cet exemple, e et L sont constantes, et r et h sont des variables de conception. Selon la formulation adaptée l'optimisation est écrite comme suit :

1- RBDO (Reliability based design optimisation)

$$\left\| \begin{array}{l} \min_{r,h} c_0 V \\ sous \\ \beta \geq 3.72 \end{array} \right. \tag{17}$$

2- HBDO (Hazard based design optimisation)

Nous plaçons le risque à 10^{-9} pour obtenir la conception intrinsèquement fiable supposant que la structure est soumise au chargement répétitif.

$$\left\| \begin{array}{l} \min_{r,h} c_0 V \\ sous \\ h \leq 10^{-9} \end{array} \right. \tag{18}$$

La résolution est obtenue avec le logiciel de MathCAD, conduisant aux résultats dans le tableau 1.

RBDO *e = 3 mm, L=0.5 m*	*HBDO* *e = 3 mm, L=0.5 m*
$r_{op} = 0.011$ m $h_{op} = 0.224$ m	$r_{op} = 0.011$ m $h_{op} = 0.231$ m
$V = 1.087 \times 10^{-4}$ m^3	$V = 1.138 \times 10^{-4}$ m^3

Tableau 1. Résultats d'optimisation selon les deux formulations RBDO et HBDO

L'augmentation du volume selon la formulation HBDO est justifié par le fait que la conception est obtenue pour une charge répétitive, tandis que la RBDO correspond à une application simple de la charge.

IV Fiabilité des structures dégradées

Alors que les études ci-dessus considèrent les deux premières phases de la vie du produit, le quatrième chapitre porte sur la phase de vieillissement, en vue de permettre la description complète de la conception pour l'ensemble du cycle de vie du produit. Pour modéliser la dégradation, deux approches se distinguent : l'approche statistique et l'approche physique. Cette dernière permet de considérer soit la marge instantanée, soit la marge cumulée. Alors que dans le premier cas, la résistance diminue, indépendamment des charges appliquées, augmentant ainsi la probabilité de défaillance, le second cas correspond au cumul des dommages jusqu'à atteindre la limite admissible. La figure 11 situe ce travail dans le contexte général de la méthode contrainte-résistance.

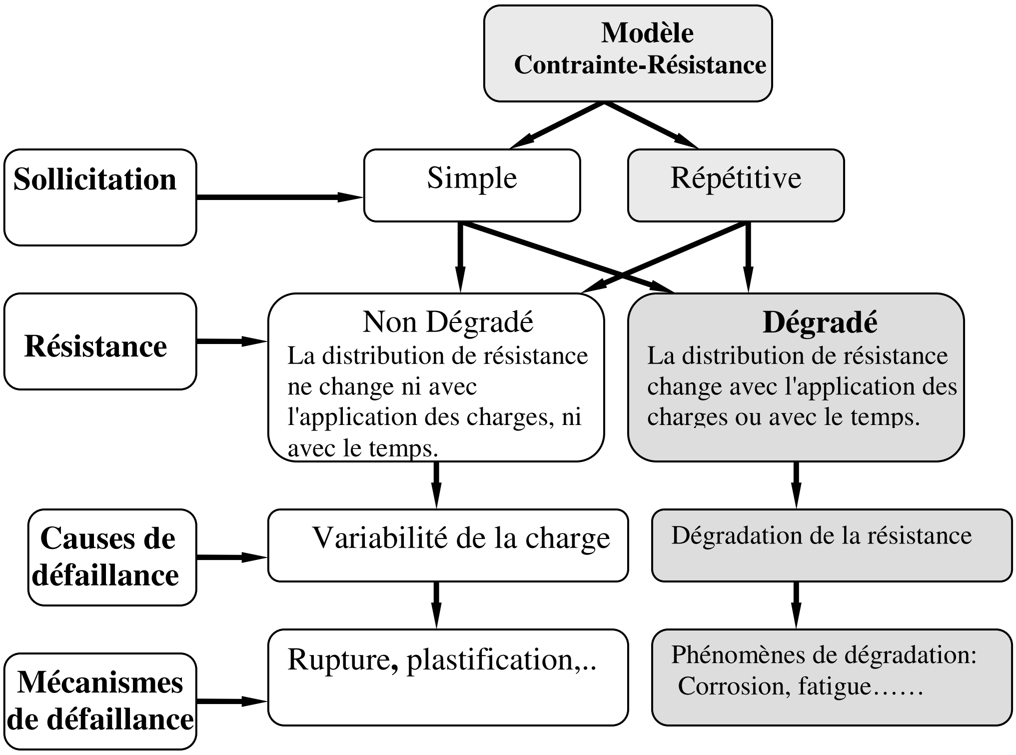


Figure 11. Classification du problème de dégradation dans le modèle contrainte-résistance.

D'une façon générale, la représentation de la dégradation peut être considérée en modifiant le modèle d'interférence pour inclure le processus de dommages tel que l'usure, l'érosion, la corrosion, le fluage et la fatigue. La densité de probabilité de la dégradation peut être donnée sous la forme :

$$f_R(.) = \text{function}\,[f_{R_0}(.), f_S(.), n] \tag{19}$$

où $f_{R_0}(.)$ est la distribution initiale de la résistance, $f_S(.)$ est la distribution de la sollicitation et n est le nombre d'applications de la sollicitation. Le processus de dégradation est limité par la distribution du seuil de dommage, défini par la limite de résistance ou de fatigue par exemple. Dans le cas de la fatigue, la fonction de résistance est représentée par la courbe *S-N*.

Selon Carter [Carter-97], parmi d'autres, la dégradation en termes d'âge d'un composant (figure 12).

$$N = \frac{1}{\int_0^\infty \frac{S(s)}{\xi(s-z)} ds} \tag{20}$$

où *S(.)* est le PDF de sollicitation, $\xi(s-z)$ est l'équation de la courbe S-$N_{F\%}$ au F% probabilité de défaillance (figure 12).

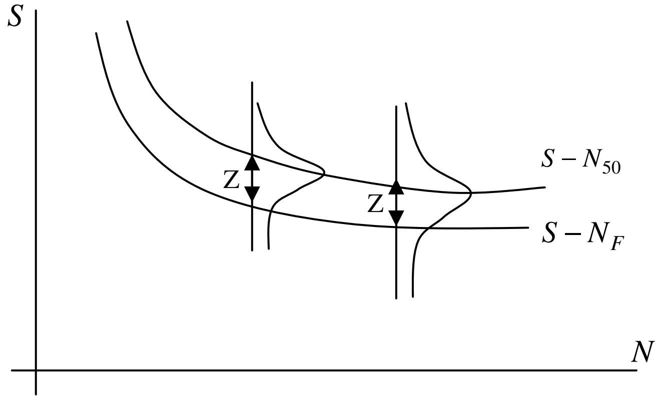


Figure 12. Courbe S-$N_{F\%}$ au F% probabilité de défaillance

La relation entre la limite d'endurance *E* et le nombre de cycles *N* est bijective. L'égalité des probabilités imposent la règle de transformation entre le seuil de dommage *E* et le nombre de cycles *N*, sous la forme :

$$f_E(s)\, ds = f(N)\, dN \tag{21}$$

où *f (N)* est la fonction de densité de probabilité de la durée de vie et $f_E(s)$ est celle de la limite d'endurance. Pour la durée de vie, la fonction de répartition et de hasard s'écrivent : $F(N) = \int_0^N f(N)dN$ et $h(N) = \frac{f(N)}{1-F(N)}$, respectivement.

IV.1 Exemple numérique

Un engrange dans une boîte de vitesse est composé d'un matériau de contrainte moyenne à la rupture de 1080 MPa et de coefficient de variation $c_R = 0.05$. La sollicitation appliquée est normale de moyenne de 1000 MPa et de coefficient de variation $c_s = 0.2$. Pour le mécanisme de fatigue, la courbe S-N de dimensionnement est définie à un fractile de 10% à partir de cinq essais à deux niveaux de sollicitation.

Niveau de sollicitation	Millions d'application de la charge par dent				
762 MPa	0.677	10.83	0.533	2.30	0.642
1272 MPa	0.23	0.279	0.274	0.335	0.392

Les données obtenues ont une distribution de Weibull avec une constante de localisation $\gamma = 0.2277x10^6$, un paramètre d'échelle $\eta = 0.2211x10^6$ et un paramètre de forme $\beta_W = 0.506$.

La courbe médiane *S-N₅₀* est ainsi estimée par :

$$N_{50} = 1.767 \times 10^{25} . S^{-6.3528}$$

Estimation de la courbe *S-N_F*

A partir de la courbe médiane, la courbe de concpetion à 10% peut être obtenue sous la forme :

$$\xi(s - z) = 1.767 \times 10^{25} (s + 77)^{-6.353} \qquad (22)$$

En substituant l'équation (22) dans (20), nous pouvons déduire le nombre de cycle à 10% au niveau de charge de conception (i.e. 1000 MPa) :

$$N_{0.10} = 1.023 \text{ x } 10^9$$

Le hasard peut ainsi être évalué pour conduire à la courbe indiquée sur la figure (13). Cette courbe montre clairement que le hasard augmente de façon très significative lorsque le produit atteint la fin de la vie utile. Il sert donc d'indicateur efficace de l'utilité du produit en service.

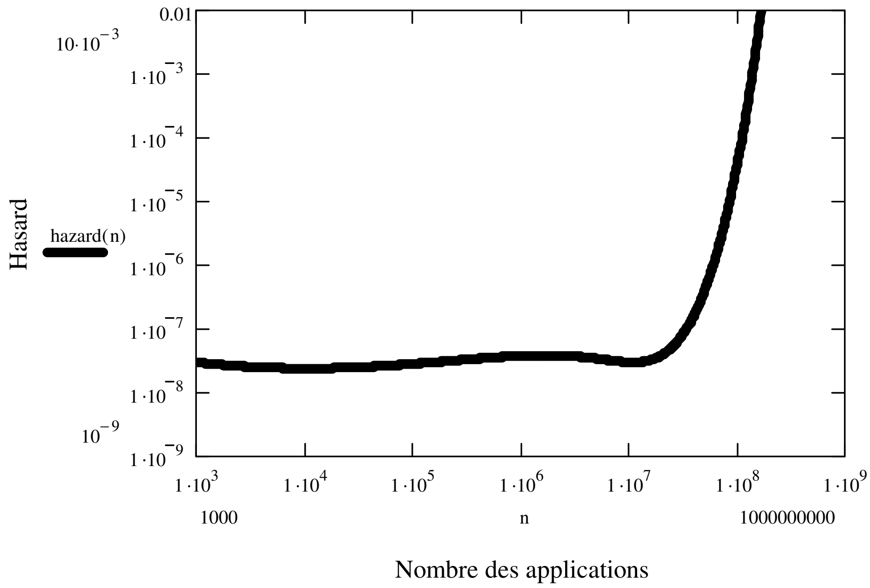


Figure 13. Fonction de hasard en fonction du nombre d'applications de charge.

V. Conclusion

Ce travail a permis d'examiner la méthodologie de conception fiable des composants mécaniques, sur la base de l'utilisation de la fonction de hasard. Après une analyse des essais de démonstration de la fiabilité débouchant sur le développement de la méthode de composition des incertitudes, il a été possible de proposer l'utilisation du hasard comme objectif de conception et de détection de la fin de la durée de vie des produits. Cette démarche constitue une base de conception intégrée du cycle de vie, dans le cadre de la maîtrise des risques industriels.

General introduction

General introduction

The process of designing and producing mechanical and structural systems consists of several stages, starting from defining the requirements and the demands throughout the life cycle, that must be supported to determine the capacity or resistance needed to fulfil the equipment mission. The reliability is the one of the most goals that manufacturers seek, while the economical aspect is a key factor and it has a great deal influence on this process. Therefore, the best design has to be carried out, in order to achieve the paradox of reliable products with minimal costs. This implies careful and exact investigation through all product life-cycle from birth to death.

Actually, product life cycle is assimilated with the humane life phases, starting from infancy where the risk of death is high to youth or stable period where the risk settled down to a stable value, and finally the aging resulting from degradation and illness at which the risk of death increases with time.

In the infancy period of products, there is a risk of mortality; this part of product life has to be carefully considered. Different strategies are concerned to get rid of it and its unlikely consequences, such as warranty and screening testing. Choosing the suitable strategy is a case of decision making related to different factors, economical one is the dominant among the others. After brief survey, in chapter I, of structural reliability theory, Chapter II, entitled "testing for reliability", presents an overview of testing procedures to achieve reliability showing its effects throughout product life cycle. The types of data obtained from testing are presented and discussed. Finally, several approaches based on structural reliability theory has been investigated for the purpose of reliability demonstration.

For the case of repetitive loading applied on non-degraded components (resistance is independent of time or number of load application), chapter III proposes to consider the hazard as a design target, leading to more robust criterion. This approach tried to build a bridge between structural reliability based on stress-resistance model which considers the failure probability as a design target and engineering reliability based on the exponential model in which the hazard is a design target. The design under negligible value of hazard gives products which are intrinsically reliable in terms of lower sensitivity towards design parameters. This means that for the case of hazard-based design, the robustness condition is satisfied, which states that designs must be insensitive to all uncontrollable parameters such as the number of load applications in the case of repetitive loading.

To complete the life-cycle of the product, chapter IV is assigned to wear-out phase. Degradation modelling considers the change of resistance distribution with time or load application (e.g. aging, cumulative damage). In this chapter, variety of models is presented to describe the degradation models from two different points of view.

The statistical model deals with the stress-resistance to cope with resistance changes by different ways, such as bi-models representing the resistance of weak components and the bulk of the other components which represents the quality control. Another similar approach in which wear out phenomena, such as fatigue and corrosion, can be modelled by distribution describing the damage threshold. In the case of fatigue; damage threshold is represented by the endurance limit distribution in S-N curve. Consequently, an estimation of life distribution and hazard function can be made.

From physical point of view, failure is defined as the first passage of stress beyond the resistance. In fact, structural reliability deals with the degradation basically in two different ways, instantaneous margin where resistance decreases with time accompanied with or without stress increase and cumulated margin which expresses the difference between the acceptable level of degradation and the cumulated degradation. Failure probability, hazard function and life probability density are the output needed for different purposes concerned by the designer. The treatment of these problems differs by the application type, and the acceptable level of risk.

Chapter I. Basic concepts of structural reliability and hazard

I.1 Introduction

In this work, the structural reliability theory based on stress-resistance model is applied in different phases of the product life cycle. Therefore, it is useful to introduce in this chapter the fundamental reliability concepts used in this thesis.

I.2 Reliability Notion

The state of the structure depends on the applied external forces, the material properties, the design models and the human factors all over the design, the realization and the operation stages.

The reliability of a structure (or a component) is the ability to carry out its design objectives during a specified reference time under a given set of conditions. Therefore, the failure of structure (or a component) is its incapacity to fulfill its objectives. In probabilistic context, the reliability is the complement to the failure probability. This is denoted by:

$$\Re = 1 - P_f \tag{I.1}$$

where $\Re$ is the reliability and P_f is the failure probability during the reference period.

I.3 Life cycle and Reliability

For a given component, the uncertainty can be described complement through the following functions: $\Re(.)$ reliability and its complement function $F(.)$ which is called cumulative distribution function CDF of failure, $f(.)$ first derivative of CDF or failure probability density function PDF, as well as the instantaneous probability of failure or hazard function, is defined by the failure probability per time unity conditioned by the survival time is greater than time *t*.

$$h(t) = \frac{f(t)}{\Re(t)} \tag{I.2}$$

Figure (I.1) gives an illustration of these basic functions. These functions show a short period of high mortality at the beginning time, followed by a weak and constant period of mortality, at the end of the structure life cycle, an increasing monotonic period of mortality is observed.

When the sample of the structures represents a population, the state functions are interpreted as functions of probability, for predetermined life time "*T*" must be achieved by the designer. The studied population gives the probability of failure by $P_f = F(T)$ and the reliability $\Re = \Re(T)$. Practically, it is difficult to determine the reliability from the observations of the population. Consequently, the theory of structural reliability makes the estimation and comparison of these probabilities possible from the structural and environmental data.

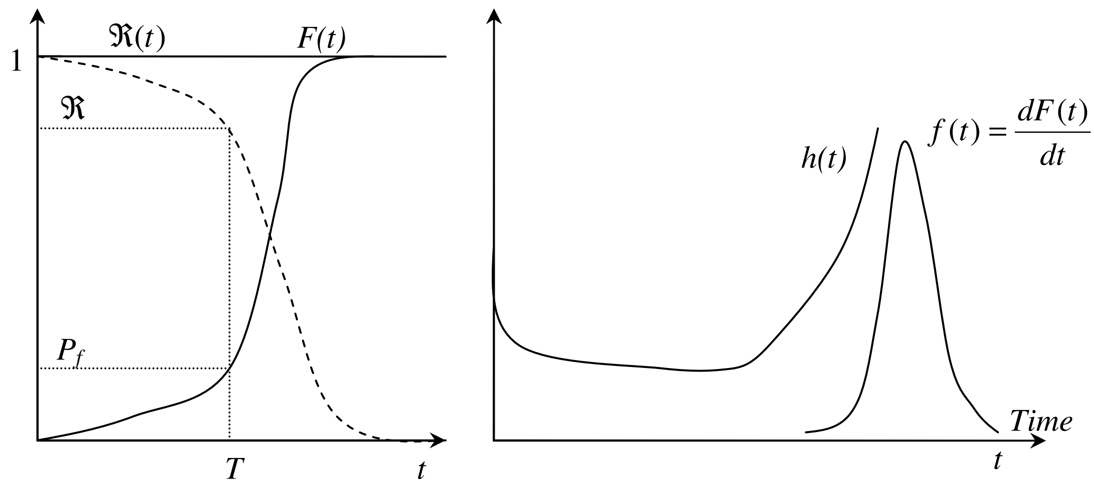


Figure I. 1functions of the structure state

I.4 Reliability and hazard rate Characterization

Let ***t*** being the time to system failure. The PDF $f_T(t)$ corresponds to the probability that failure takes place at a time between t and $t+\Delta t$, divided by Δt :

$$f_T(t)\Delta t = P[t < T \leq t + \Delta t] \tag{I.3}$$

The $F_T(t)$ indicates the probability that failure takes place at a time less or equal to *t.*

$$F_T(t) = \int_0^t f_T(z)dz = P\{T \leq t\} = P_f(t) \tag{I.4}$$

The reliability can thus be defined as the probability that a system operates without failure for a span of time *t.* This quantity is also known as cumulative distributed function CDF.

$$\Re(t) = P[T > t] = 1 - F_T(t) = 1 - P_f(t) \tag{I.5}$$

The hazard function $h(t)$ may be defined in terms of probability that the system will fail at some time interval t <***T*** < $t + \Delta t$**,** given that it has not yet failed at ***T*** = *t*. It is given by the conditional probability:

$$h(t)\Delta t = P[t < T < t + \Delta t / T > t] = \frac{P[(\mathrm{T} > t) \cap (\mathrm{T} \leq t + \Delta t)]}{P[\mathrm{T} > t]} \tag{I.6}$$

The numerator on the right-hand side is just an alternative way of writing the PDF; that is:

$$P[(\mathrm{T} > t) \cap (\mathrm{T} \leq t + \Delta t)] \equiv P[t \leq \mathrm{T} < t + \Delta t)] = f(t).\Delta t \tag{I.7}$$

The denominator of equation (I.6) is just $\Re(t)$, therefore, we get:

$$h(t)=\frac{f(t)}{\Re(t)}=\frac{1}{\Re(t)}\frac{dF(t)}{dt} \tag{I.8}$$

This quantity, is given a variety of other names [Car-86], such as ' *the failure rate*',' *the force of mortality*', from actuarial practice , or '*mortality intensity*', '*hazard*', '*hazard function*' , '*hazard rate*', '*age-specific failure rate*', '*instantaneous failure rate*', is also referred as the '*mortality rate*', '*conditional failure rate*'. The terminology Hazard is chosen in this manuscript. Actually, the practical importance of the hazard $h(t)$ is perhaps best demonstrated by the product $h(t)dt$ where dt is a small interval of time. Given that from equation (I.8):

$$f(t)=-\frac{d\Re(t)}{dt} \tag{I.9}$$

the hazard equation can be expressed in terms of reliability:

$$h(t)=\frac{dF(t)}{\Re(t)dt}=-\frac{1}{\Re(t)}\frac{d\Re(t)}{dt} \tag{I.10}$$

By integration, the reliability takes the form:

$$\Re(t)=e^{-\int_0^t h(\tau)d\tau} \tag{I.11}$$

For the case of constant hazard $h(\tau)=\lambda$ the reliability equation becomes:

$$\Re(t)=e^{-\lambda t}$$

Let us now express the hazard as a function of the number of load applications; the previous equation takes the form:

$$h(n)=\frac{F(n)-F(n-1)}{\Re(n)} \quad n\geq 2 \quad \text{[Car-97]} \tag{I.12.a}$$

or

$$h(n)=\frac{\Re(n-1)-\Re(n)}{\Re(n)} \quad n\geq 2 \tag{I.12.b}$$

It is emphasised that t or n measures the age of non-maintained part or component. Although, reliability is usually the specified target, it is expressed in terms of the cumulative failure probability *F(t)*, which should be kept very small. In many applications, the hazard (particularly when it is constant) gives a more convenient criterion. In fact, reliability varies with time or number of load applications. In this sense, the hazard must be very low in the product life, but it increases sharply when wear is set in, at this stage practical design criterion is the time or the number of loads to reach a specified percentage of failures, after which the product cannot be considered, fit its purpose. This percentage is often written as B_F in the literature because it was historically introduced in connection with bearing lives. Usually, the unit of hazard is failures per unit time or number of cycles, such as failures per hour or failures per mile. In high-reliability electronics applications, FIT (failures in time) is the commonly used unit, where 1 FIT equals to 10^{-9} failures per hour. In the automotive industry, the unit "failures per 1000 vehicles per month" is often used or PPM (part per million).

Probably the most used parameter to characterize reliability is the Mean Time To Failure (MTTF), which is the expected value **E[T]** of the failure time.

$$\begin{aligned} MTTF &= \int_0^{\infty} t.f_T(t).dt = \int_0^{\infty} \Re(t).dt \\ &= E[T] \end{aligned} \tag{I.13}$$

I.4.1 Cumulative Hazard function

The cumulative hazard function, denoted $H(t)$, is defined as:

$$H(t) = \int_0^{t} h(\tau)d\tau \tag{I.14}$$

For the exponential distribution, we have $H(t) = ht$, and hence: $\Re(t) = \exp(-H(t))$. If $H(t)$ is very small, a Taylor series expansion results in the following approximation:

$$\Re(t) \approx 1 - H(t) \tag{I.15}$$

$H(t)$ is a non-decreasing function as depicted in figure(I.2) for decreasing hazard $\mathbf{D_h}$ (convex shape), constant hazard $\mathbf{C_h}$ (flat shape) and increasing hazard $\mathbf{I_h}$ (concave shape), respectively.

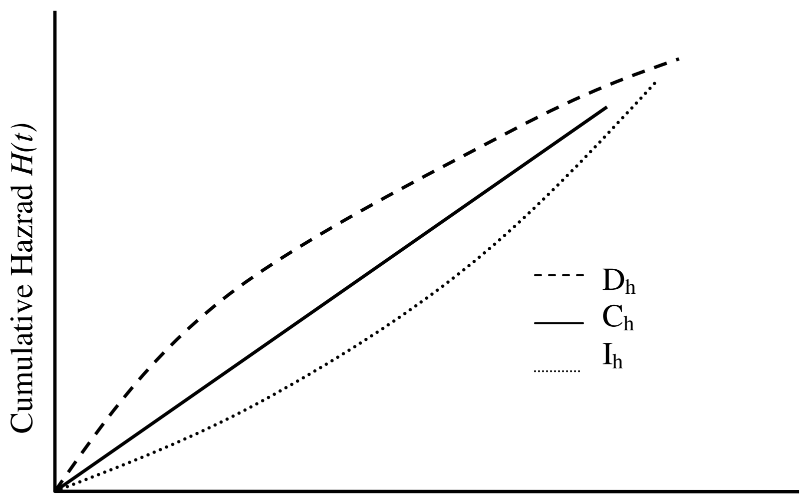


Figure I. 2 Cumulative hazard functions corresponding to D_h, C_h and I_h

I.5 Reliability levels

Reliability is classified in different levels [Mad-86], according to the available information about the studied structure. Four levels are distinguished:

- **Level I methods**: the methods employ one characteristic value of each uncertain parameter. Load and resistance factor formats including the allowable stress format are examples of it.
- **Level II methods,** the methods employ two characteristic values of each uncertain parameter (commonly, the mean and variance). Reliability index methods belong to this level.
- **Level III methods:** these methods employ failure probability as a measure, and therefore require the knowledge of the joint distribution of all uncertain parameters.
- **Level IV methods:** they compare a structural prospect with reference prospect according to the principles of engineering economic analysis under uncertainty, considering cost benefits, of construction, maintenance, repair, consequences of failure and interest on capital…etc**.** Such methods are appropriate for structures that are of major economic importance if the prospect of loss of life and cultural values are minor. Highway bridges, transmission towers, nuclear power plant structures are examples for this level.

We can find some reliability method which combine or mix two or more of the levels mentioned above.

I.6 Resistance –Stress interference reliability model

The simplest situation is that when the reliability of a component is determined by two random and independent variables: load variable S and resistance variable R. The failure is observed when the load exceeds the resistance (figure I.3).

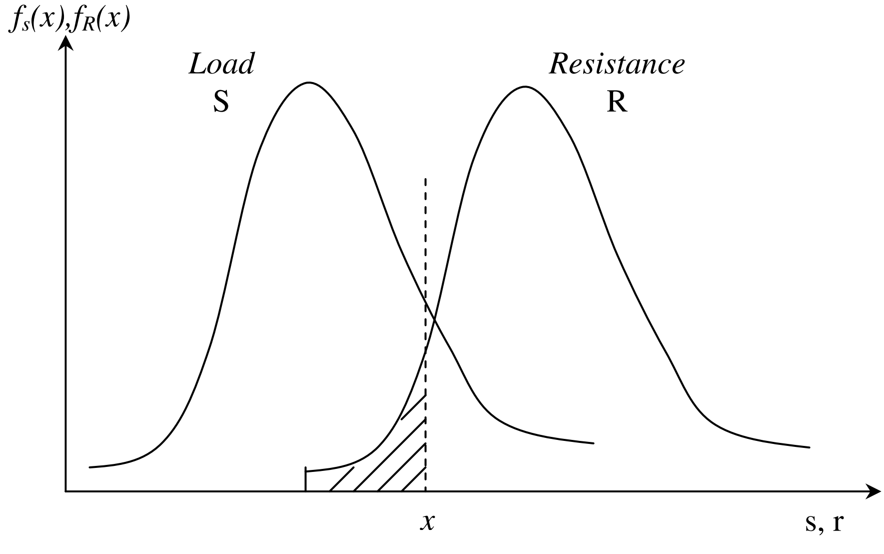


Figure I. 3 stress and resistance distributions

The failure probability P_f is given by the probability of reaching a certain load level under the condition that resistance is lower than this level. For the case of independent variables, it is expressed by:

$$P_f = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx \tag{I.16}$$

where $F_R(.)$ is the cumulative distribution function of the resistance and $f_S(.)$ is the probability density of the load.

I.6.1 General case of structural reliability

The first step in the evaluation of reliability consists in identifying a certain number of variables X_i for which the uncertainty has significant influence. These variables can be the applied loads (waves or wind), the properties of used materials (e.g. yield stress, Young's modulus or Poisson's ratio) or the dimensional characteristics (e.g. length or moment of inertia of the members). These variables are called *basic design variables*. They could be modeled random variables or stochastic processes.

The space of the basic variables is divided into two regions, denoted the failure region and the reliability region (figure I.4). The separating surface of the two regions is called failure surface, noted $G_f(\{x\})$.

$$G_f(\{x\}) = G_f(x_1, \ldots\ldots, x_n) = 0 \tag{I.17}$$

where $G_f(\{x\})$ is called the failure function or the limit state function. It is defined in a way that if it is positive, the state of the component is placed in the safe domain. In the contrary case, the combination of the basic variables indicates the failure state.

$$G(\{x\}) > 0 \qquad \text{if} \qquad \{x\} \in \Omega_s \tag{I.18.a}$$

$$G(\{x\}) \leq 0 \qquad \text{if} \qquad \{x\} \in \Omega_f \tag{I.18.b}$$

where Ω_S and Ω_f are respectively the reliability and failure domains. It results from equation (I.18) that the case where $G(\{x\}) = G_f(\{x\})$ belongs to the failure domain, it is null measure.

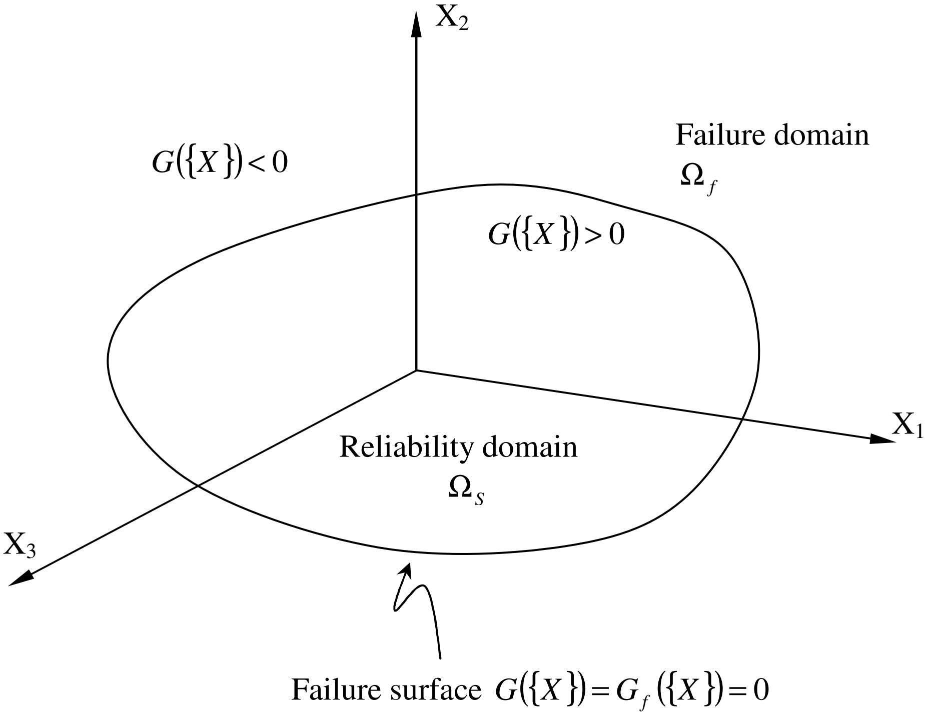


Figure I. 4 Reliability and failure domains

I.6.2 Cornell reliability index

The reliability index is a measure to estimate the component reliability. **Rajanitzyne and Cornell** [Mau-96] defined the reliability index β_C as following:

$$\beta_C = \frac{m_G}{\sigma_G} \tag{I.19}$$

where G is the safety or reliability margin, m_G is its mean value and σ_G is its standard deviation. This definition is illustrated geometrically in figure I.5. The failure surface becomes in the case of one dimensional a point *(g=0)*. The interpretation of Cornell index β_C assumes that the distance between the expectation position m_G and the failure surface represents a good measure of reliability. This distance is measured with the scale parameter of uncertainty σ_G.

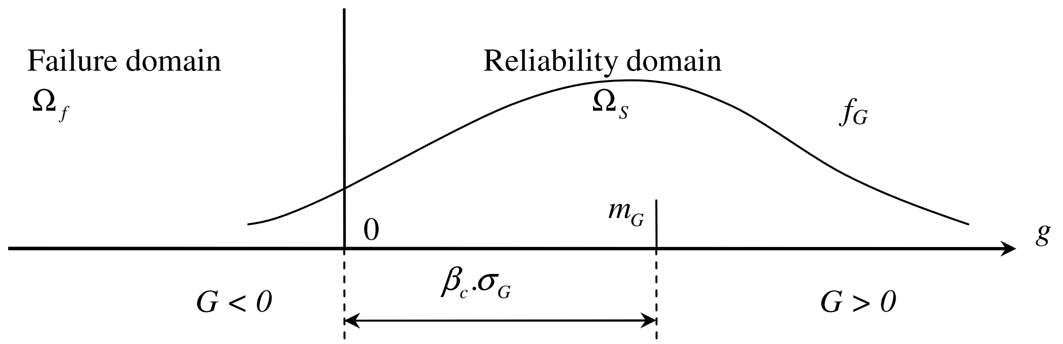


Figure I. 5 One-dimensional Cornell index

Consider the case of two independent Gaussian variables: *R* for resistance and *S* for stress. The safety margin is written:

$$G(r,s) = r - s \tag{I.20}$$

where *r* and *s* are the realization of *R* and *S*, respectively. The Cornell reliability index can therefore be written as:

$$\beta = \frac{m_G}{\sigma_G} = \frac{m_R - m_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{I.21}$$

where m_R, m_S are mean values of resistance and stress independently and σ_R, σ_S are the standard deviations of resistance and stress respectively. This represents a particular case of linear reliability margin. In the case of multi-dimensional, it is expressed as a function of random variables under the form:

$$G = a_0 + a_1 X_1 + ... + a_n X_n \tag{I.22}$$

where a_i are constants and X_i are normal variables. The reliability index is:

$$\beta = \frac{m_G}{\sigma_G} \tag{I.23}$$

with

$$m_G = a_0 + a_1 m_{X_i} + ... + a_n m_{X_n} \tag{I.24}$$

and:

$$\sigma_G^2 = a_1^2 \sigma_{X_1}^2 + ...a_n^2 \sigma_{X_n}^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \rho_{X_i X_j} a_i a_j \sigma_{X_i} \sigma_{X_j} \tag{I.25}$$

where $m_{X_i}, \sigma_{X_i}^2$ are the mean value and the variance of the variable X_i, and the correlation coefficient:

$$\rho_{X_i X_j} = \frac{Cov[X_i, X_j]}{\sigma_{X_i}\sigma_{X_j}} \tag{I.26}$$

where cov[X_i , X_j] is the covariance between X_i and X_j. Since, the random variables are normally distributed, it can be demonstrated that the failure probability can be directly linked to the reliability index by:

$$P_f = P(G \le 0) = \Phi\left(\frac{0-\mu_G}{\sigma_G}\right) = \Phi(-\beta_C) \tag{I.27}$$

where $P(.)$ is the failure probability operator and $\Phi(.)$ is the standard Gaussian CDF. This equation shows the simplicity obtained induced by safety margin linearity.

I.6.3 Hasofer and Lind reliability index

Cornell index is limited to hyper-plane failure surfaces with Gaussian variables. In the case of nonlinear failure surfaces, the calculation of Cornell index implies the linearization of this surface. If the variables are not normal, the reliability index value varies according the linearization point and to the considered space of variables. To assure the invariance in the calculation of β, Hasofer and Lind [Has-74] have developed a more general method. Reliability index has a geometrical interpretation such the minimal distance between the origin and the failure surface inside the normal space. Hasofer and Lind have proved that the linearization must be done in the most probable failure point. This implies a transformation of the physical space X_i into the normalized space U_i. Thus we have:

$$\{U\} = T(\{U\})$$

where $T(.)$ is the probabilistic transformation function, when applying this transformation, the median value in the x-space becomes the origin point in the u- space. Also the $G_f(\{X\})$ is transformed to a corresponding failure surface $G'_f(\{U\})$. The distance to the failure surface can be measured by Hasfer-Lind reliability index, computed by solving the optimisation problem.

$$\beta(\{u\}) = \min_u \left(\{u\}^t\{u\}\right)^{1/2} \text{ under } \{u\} \in G'_f(\{u\})$$

The solution of this equation is obtained at the point $\{x^*\}$. This point called design point and interpreted as the most probable failure point. The determination of the position for this point considered as a minimization problem between this point and the origin. This minimization is conditioned by belonging to the failure surface [Shi-83].

$$\beta = \min\left(\sum_{i=1}^{n} u_i^2\right) \text{ with } G(u_i) = 0 \tag{I.27}$$

This procedure permits a rigorous and unique determination of reliability index. The determination of Hasofer Lind index is illustrated in (figure I.6).

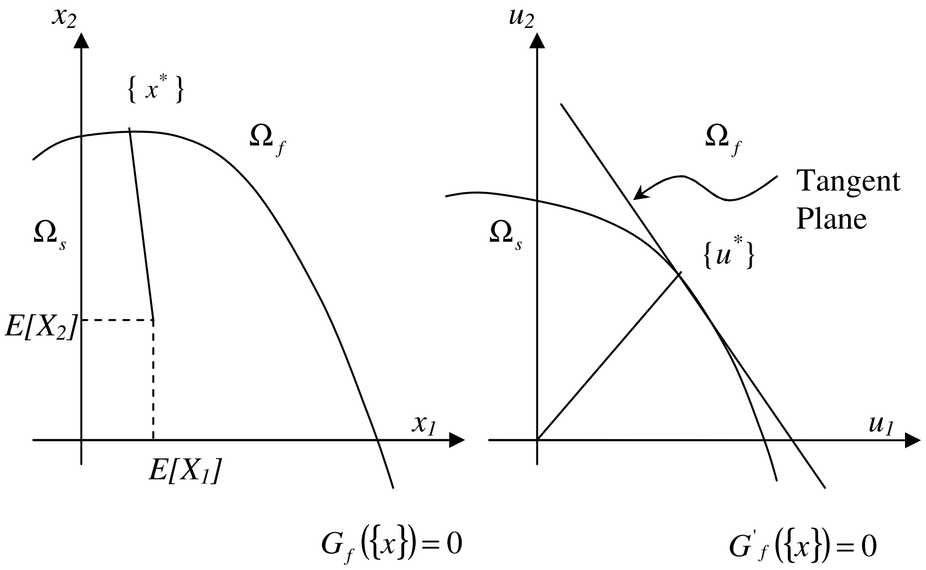


Figure I. 6Hasofer Lind index

The Cornell index β_c and Hasofer Lind one β_{HL} are coincident when the failure surface becomes hyper-plane in the standard space. Therefore, Cornell index is a particular case of Hasofer-Lind index.

I.6.4 Gaussian non-correlated case

In this case of independent normal variables, the transformation of physical space into normal space is:

$$X_i \xrightarrow{T} U_i = \frac{X_i - m_{X_i}}{\sigma_{X_i}} \tag{I.28}$$

It is a linear transformation from Gaussian distribution X_i into standard Gaussian distribution (i.e. with a zero mean value and unit standard deviation $N[0,1]$). The limit state function $G(R,S) = R - S$ becomes:

$$H(U_R, U_S) = m_R + \sigma_R U_R - m_S - \sigma_S U_S \tag{I.29}$$

The minimum distance between the origin O and the failure surface $H(u_R, u_S) = 0$ is given by:

$$\beta = \frac{m_R - m_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{I.30}$$

We can rearrange this equation $H(U_R, U_S) = 0$ by dividing $\sigma_G = \sqrt{\sigma_R^2 + \sigma_S^2}$:

$$\frac{m_{R_1} - m_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} + \frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_S^2}} U_1 - \frac{\sigma_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} U_2 = 0$$

$$\beta + \alpha_R U_1 - \alpha_S U_2 = 0$$

where α_R and α_S are the direction cosines of resistance and load respectively; they represent the influence of the corresponding random variables on the limit state function equation. In fact, for this particular case the directional cosine of loading is called the loading roughness [Car-97].

I.6.4.1 Loading roughness

For the case of safety margin $G = R - S$ with normally distributed independent variables, the β reliability index has been shown to be expressed as a normalized safety margin:

$$\beta = \frac{m_s - m_R}{\sqrt{\sigma_S^2 + \sigma_R^2}}$$

In this expression Carter [Car-86] has defined the loading roughness; as non-dimensional parameter representing the load dispersion as a fraction of the safety margin dispersion.

$$\text{Loading roughness} = LR = \alpha_s = \frac{\sigma_S}{\sqrt{\sigma_S^2 + \sigma_R^2}} \tag{I.30}$$

The *LR* is the direction cosine of stress in the failure surface equation. Therefore, it varies between *0* and *1*; *0* represents the case of deterministic load and *1* represents the case of deterministic resistance.
According to *LR* values, different types of loading are distinguished (figure I.7).

- *Smooth load*: corresponds to low values of *LR*, in this case the load distribution is confined to a small range, but that of the resistance is much wider. Generally, the electrical components and some of the mechanical components have limited and controlled loads. Typical mechanical example for this type of load is the gun.

- *Rough load*: the case is vice-versa between load and resistance distributions load dispersion is much higher than material dispersion. Usually, the mechanical equipments are subjected to much rougher loading, because of the difficulty of controlling the environment.

- *Medium load:* this case corresponds to the situation in-between for both load and resistance.

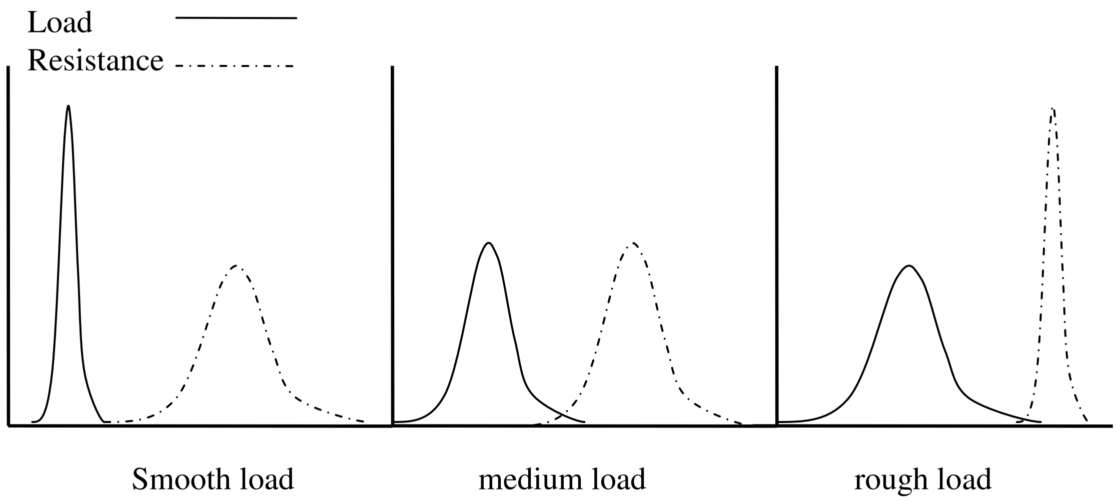


Figure I. 7 Types of load roughness representation

I.6.4.2 Loading roughness importance

According to the roughness of loading, the system reliability calculation differs completely. To explain the idea, let us consider a mechanical system composed of *n* items in series. Every component in this series system has the reliability $\Re$, we are going to calculate the system reliability or *n* components in series in two extreme situations (figure I.8).

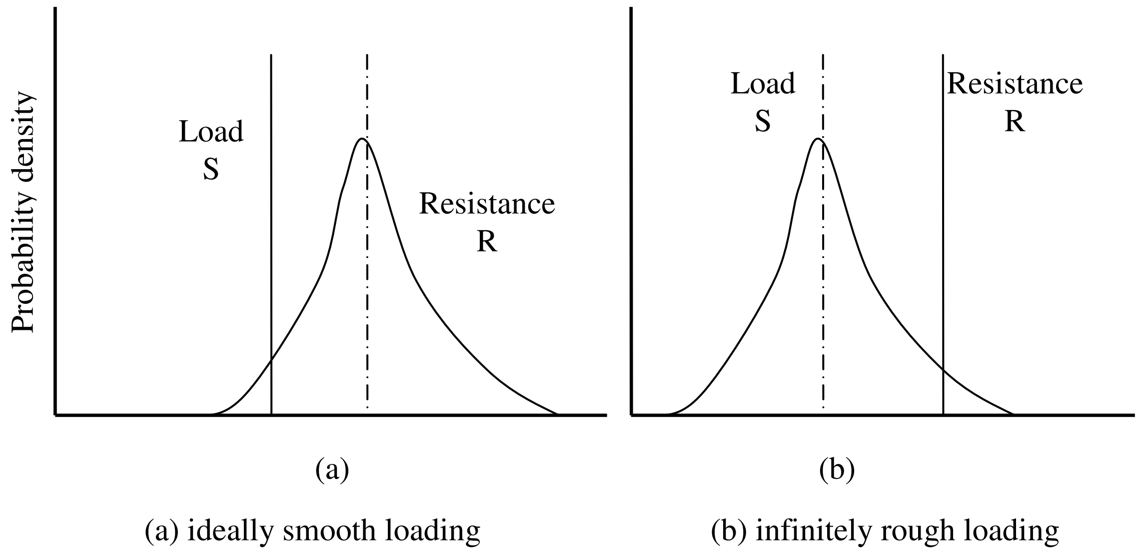


Figure I. 8 Extreme loading conditions

For the situation (a), the reliability of the first component in the system is $\Re_1$, which is the probability that its resistance is greater than the unique load S. Similarly the reliability of each of the other components is $\Re_1 = \Re_i$. Hence, if the unique load S is applied to the series system the probability of the first component can withstand the load is $\Re_1$; the probability that both the first and the second can withstand the load $\Re^2$, and so on, leading to the general case of *n* components for which the reliability will be given by:

$$\Re_{sys} = \Re_i^{\,n} \tag{I.31}$$

where $\Re_{sys}$ is the total system reliability. Let us consider the situation of rough load. The reliability of the first component in the system is $\Re_1$, as before. Now if the randomly selected load is applied to the series system, the probability that the system can withstand the load is $\Re_i$. However, since all the components have the same unique resistance $\Re_i$, if the first component withstands the load, we are 100 per cent certain that all other components can withstand the same load. Hence, applying the product rule gives that the reliability of the series system is

$$\Re_{sys} = \Re_i \text{x } (1)^{\text{n-1}} = \Re \tag{I.32}$$

In the field of electrical products, the load is very smooth and the situation applied roughly to the case (a); thus the multiplication rule is applicable. However, it is not the case for mechanical products because of the high degree of load roughness. Figure I.9 adapted from [Car-86], shows the overall reliability of *n* components for series system versus the number of components for the two mentioned cases. It will be seen that for smooth loading in which the load is well defined, the overall reliability drops rapidly with an increase in the number of components, this is confirmed by equation (I.31), whereas for rough loading, in which the scatter in load is great in comparison with resistance, the curve is indeed much closer to the approximation $\Re_{tot}$ = constant $= \Re_i$. These are the extreme examples, and a medium loading lying between (1) and (2) figure I.9.

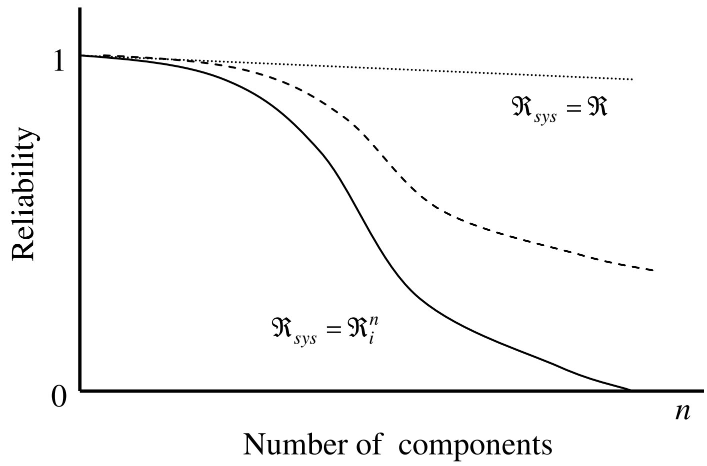


Figure I. 9 reliability variation with number of components

In fact, in the case of non-degraded systems (invariant resistance distribution with time) loading roughness determines their reliabilities. Therefore, modelling the load becomes very important task in this case.

I.7 Product life cycle and hazard function "bathtub curve"

The hazard curve has usually the general characteristics of a "bathtub "such as shown in Figure (I.10). In fact, the bathtub curve is a characteristic of living creatures. Comparisons of human mortality and engineering failures give the three broad classes of failures.
The first period of life is a region of high but decreasing hazard. This is referred as the period of infant mortality. Defective pieces of equipment, subjected to failure because they were not designed, manufactured or constructed properly, causes the high initial hazard of engineering devices. Early failures in engineering are assimilated to "product noise" quality loss in the Taguchi methodology [Lew-94]. It is preferred to eliminate such failures through design and production quality control measures such as environmental stress screening and in proof–testing (wear-in).

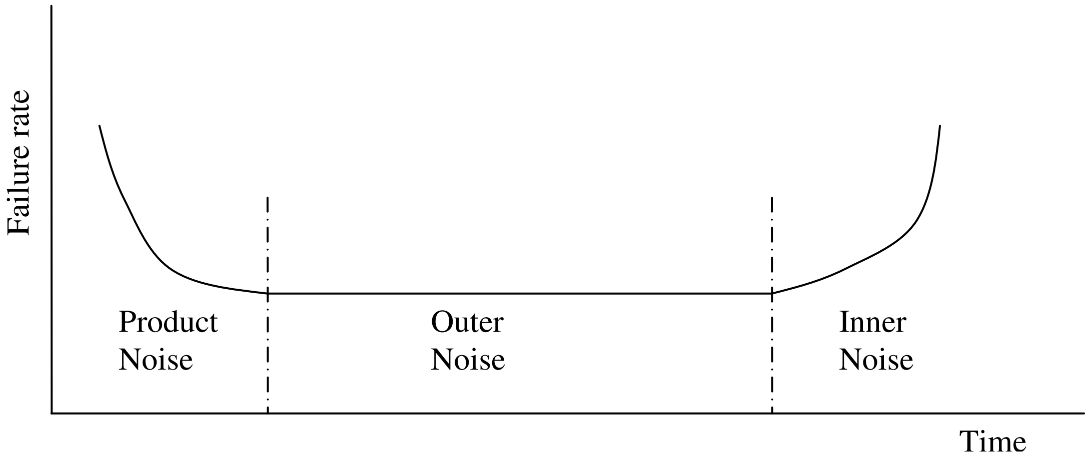


Figure I. 10 A "bathtub" curve representing a time-dependant hazard

The middle part of the bathtub curve is referred to as the "useful life". The flat behaviour is characteristic of failure caused by random events "random failures". Earthquakes, power surges, vibration, mechanical impact, temperature fluctuations and moisture variation are some common causes. In Taguchi quality methodology such loads are referred to as "outer noise". Random failures can be reduced by improving designs: making them more robust with respect to the environment to which they are subjected. On the right of the bathtub curve is a region of increasing failure rates. During this period of time, ageing failures become dominant by cumulative effects such as corrosion, fatigue cracking, and degradation of materials. Design with more durable components and materials, inspection and preventive maintenance are the approaches to produce longer-life products. In Taguchi methodology, the causes of deterioration are referred as "inner noise".
Many products do not illustrate a complete bathtub curve. Instead, they have one or two segments of the curve. For example, most of mechanical parts are dominated by wear-out mechanisms and thus have an increasing hazard rate. Some components exhibit a decreasing hazard in the early period, followed by an increasing hazard rate, without constant failure stage.
Whereas, the electrical components exhibit almost constant hazard, expect for two small periods of infant mortality and aging phases (figure I.11).

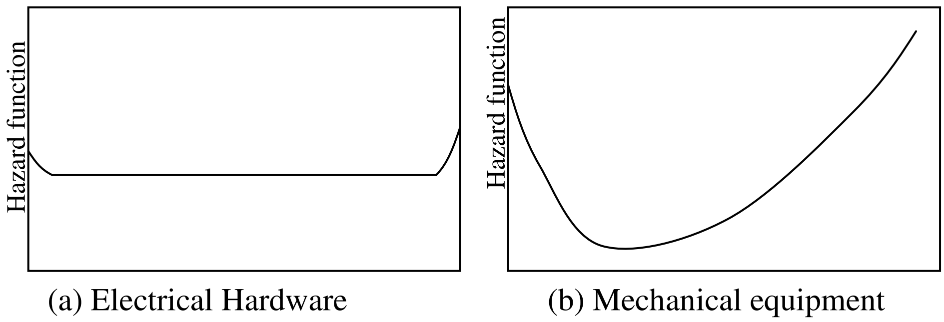


Figure I. 11 Hazard function representation for (a) electrical and (b) mechanical equipments [Lew-94]

Besides, in some cases it is noticed that some mechanical components have a roller-coaster in wear-out phase, due to some internal defects [Car-97]. This can be justified by the propagation of a physically small defects starting from minimum defect size and terminating at large defect size inducing failure. This implies that the products have initial defects will fail sooner (figure I.12). This is shown in [Bom-69] as a result from fatigue tests at constant strips from aero-engine compressor disc material.

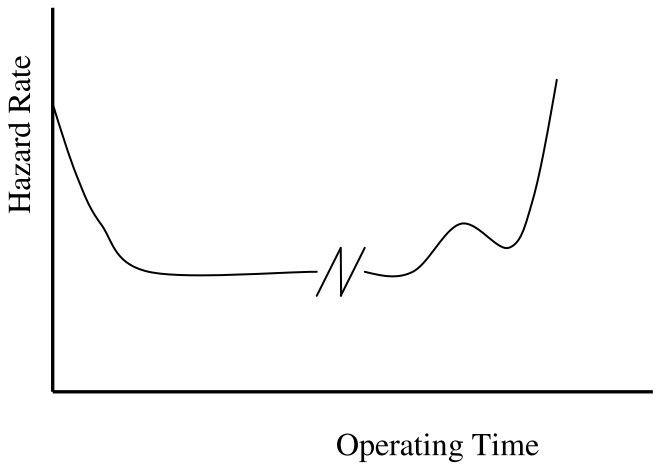


Figure I. 12 The roller coaster curve in wear out phase [Bom-69]

The same phenomenon appeared in burn-in phase for electrical components [Won-90, Eng-95] as shown in figure (I.12). According to the authors mentioned before this type of fluctuating hazard is due to latent failure, it happened when the internal or external stresses exceed the design resistance, there is often “jump” in the hazard curve as the failures are exposed as shown in figure (I.13). As an example, this curve was noticed in the hazard rate curve resultant from testing a group of 23 satellites [Ham-88].

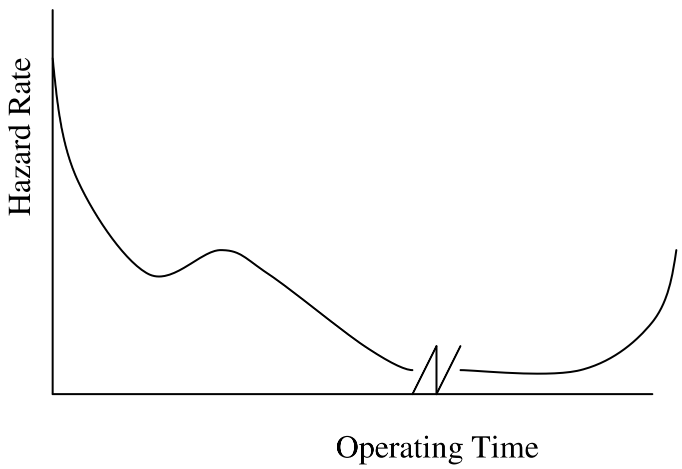


Figure I. 13The roller coaster curve in wear in phase

I.8 Conclusion

In this chapter, we gave a brief review of the fundamental principles of the structural reliability theory, with a special attention to stress-resistance model, as well as the bathtub curve. These basics will be recalled for the developments carried out in this thesis. The review shows the importance of the statistical data for the design of reliable products. It shows also the importance of load roughness consideration in the design model. In addition, the wear out phenomenon is mandatory for life lifetime management of mechanical systems.

Chapter II. Testing for reliability

II.1 Introduction

Global competition has placed great pressure on manufacturers to deliver products with higher reliability at lower cost and in less time. The new challenges have motivated manufacturers to develop and deploy effective reliability programs. In fact, an effective reliability program consists of a series of reliability tasks to be implemented throughout the product life cycle, including product planning, design and development, verification and validation, production, field deployment, and disposal (figure II.1) [Yan-07]. The reliability activities are not independent exercises; rather, they should be integrated into engineering projects in each stage of the life cycle and assist successful completion of the projects.

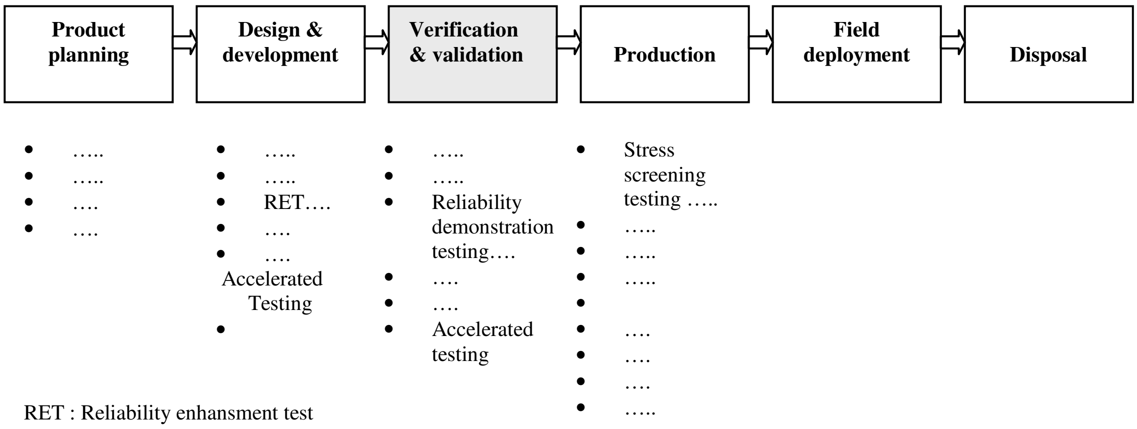


Figure II. 1. Reliability tasks for typical product life cycle [Yan-07]

These tasks include different types of reliability testing that are considered as the cornerstone of a reliability engineering program. Generally, the purpose of reliability testing is to discover potential problems with the design as early as possible and, ultimately, provide confidence that the system meets its reliability requirements. Therefore, it provides the most detailed form of reliability data. The type of reliability testing that product undergoes is different, depending on the points of its life cycle, but the overriding goal is to ensure that data, generated from all or most of the tests, can characterize the product reliability at different points of its life cycle. For this reason, reliability specifications and standard definitions of failure are up-front requirements to implement reliability tests. These tests may be performed at various levels. For example complex systems may be tested at component, unit, assembly, subsystem and system levels. However, testing reliability requirement is problematic for several reasons: a single test is insufficient to generate useful statistical data, multiple tests or long-duration tests are usually very expensive, and some tests are simply impractical.

II.2 Reliability test description

In general, to perform a reliability test, different information must be determined (figure II.2), basically the applied stresses, their nature (i.e. environmental such as temperature, humidity, or mechanical such as force, torque, others), the application method (constant, time dependent, single or multiple), the stress level within specification, the optimal operating, the degradation or destruction zones (figure II.3). These zones are bounded by specification limits and represent the definition of the product between the customer and supplier, **The Operating limits,** represent boundaries on product operability, beyond which a product will cease to function properly. However, once the elevated stresses are reduced the product will function again; **the destruction limits** denotes the boundaries beyond which irreversible damage may occur to the product. Sample size or test time, these two parameters depend on the type of test for example in Bogey testing, the sample size must be determined before commencing test; whereas it is not the case in sequential and stress resistance tests. Finally, the different types of data obtained (complete or censored) are treated in different techniques for estimating the life distribution parameters.

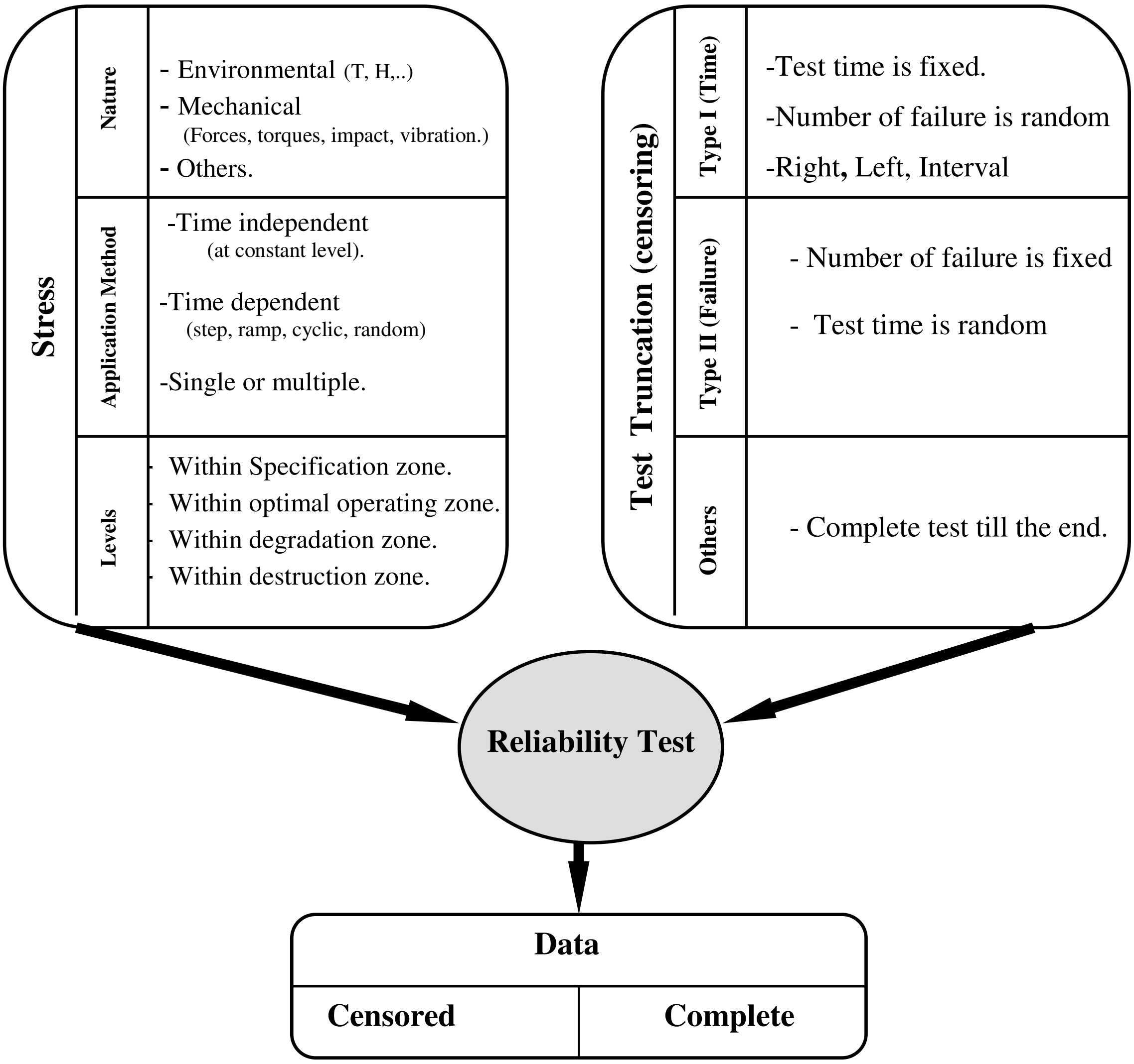


Figure II. 2. General reliability test description

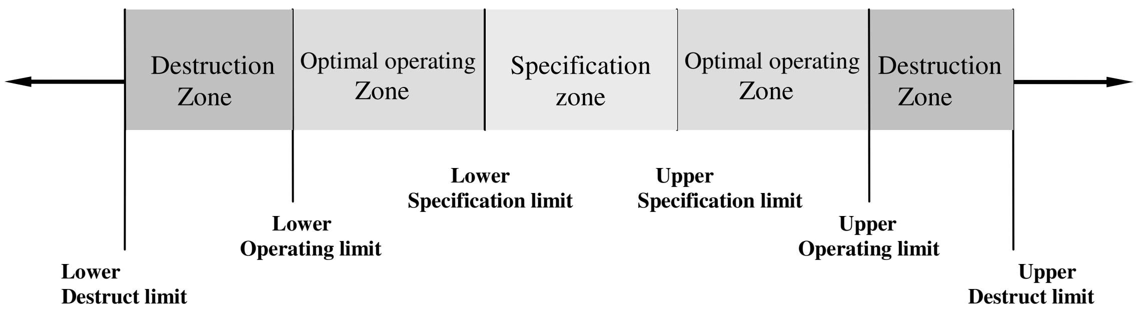


Figure II. 3. Typical stress range for a component, product or system [Wass-03]

II.3 Impact of different types of tests on product reliability

Three basic types of tests have considerable effects on life span tests:

- **Environmental stress screening (ESS)** is aimed at exposing infant mortality failures which would otherwise occur early in the life of the product (figure II.4).
- **Reliability enhancement tests (RET)** are conducted to find early failures related to the product design, but it is also used to determine the robustness of the product with respect to random failures along the useful life period.
- **Accelerated life tests (ALT**) are aimed at finding ***how, when*** and ***why*** wear out failures occur in the product.

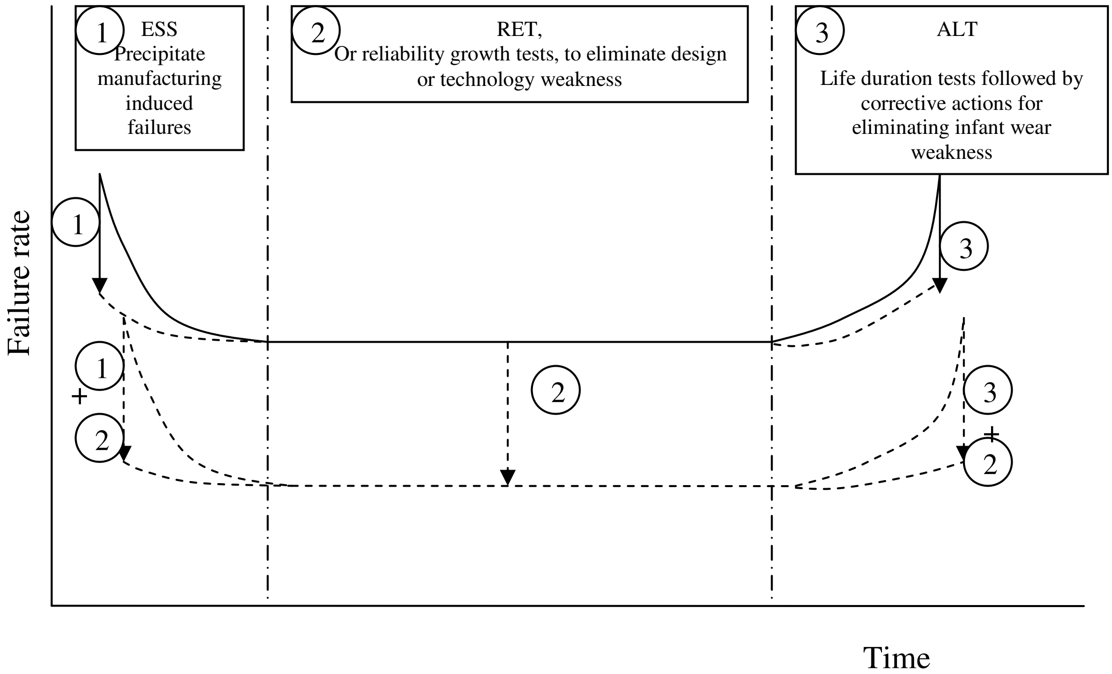


Figure II. 4. Impact of different types of tests on product reliability

Two out of these three types of accelerated tests (i.e. ESS and RET) are **on-line processes** and the third one (i.e. ALT) is an **off-line process. On-line** processes are those which are part of the design and production cycles. They are conducted with samples of the actual product. **Off-line** processes are those which are not part of the design and production cycles. They are usually not conducted on actual product samples, but on generic samples representing materials, the components or the processes used to manufacture the product. These types of tests will be described in the following subsections.

II.3.1. Environmental Stress Screening

Environmental Stress Screening (ESS) is useful in minimizing the early failures of manufactured products by screening latent defects. ESS is one of the most used reliability screening tests. Its purpose is to precipitate latent defects, which are detectable only with the application of stress. The defects are ideally those introduced into the product during manufacturing, since design-related defects should have been detected and eliminated by reliability enhancement testing during the design phase. ESS is effective only for a product with infant mortality region, which is indicated by degreasing initial failure rate in (figure II.4). ESS should be based on an understanding of the potential types of latent defects in the product, the failure mechanisms and the stresses that generate them. ESS conditions are set up to precipitate those defects and the results are used to determine their causes to undergo preventive actions. There are different screening techniques being currently implemented in industry such as burn-in, environmental stress screening (ESS) and highly accelerated stress screening (HASS). The literature for this kind of tests is very rich for electrical components whereas it is not like that for mechanical ones [Kec-99, Mil- STD -781, MIL-STD-883F]. For example, the burn-in strategies for the microcircuits specified in MIL-STD-883F (U.S. DoD, 2004) require electrically loading the devices at a minimum of 125°C for 168 hours. The burn-in strategies are effective in weeding out surface and metallization defects and weak bonds, [Jen-82] and [Kuo-98] describe burn-in techniques. Similar to burn-in, ESS is also a screening method that subjects all products to an elevated stress level for a predetermined duration. ESS differs from burn-in in that it exposes products to environmental stresses outside the specification limits. The most commonly used stresses are thermal cycling, random vibration, power cycling, temperature and humidity. In applications, the combination of two or more stresses is often used to enhance screening effectiveness. MIL-HDBK-344A (U.S. DoD, 1993) well documents the techniques for planning and evaluating ESS programs for military electronic products. A HASS is a more stressful ESS. In a HASS, the applied stresses may not necessarily be the ones that would be experienced in the field [Hob-00].

II.3.2. Reliability Enhancement (growth) Testing (RET)

The purpose of RET is to determine the types and levels of environmental stresses producing failures in the product, given that there are no defects in the materials and components used in manufacturing. In this sense, RET is a type of inspection test for the product design processes. Because RET is not directed toward finding infant defects, the sample size can be very small. The ideal time to conduct RET is at the end of the design cycle, when the expected design, materials, components and manufacturing processes are available, and production has not yet begun. RET is not a qualification test, since its purpose is to find weak spots in design and correct them before production begins.

RET is usually conducted by applying the expected environmental and operating stresses (singly, sequentially or simultaneously) initially at low levels, and then increasing them in steps until one of the following three events occur:

- All (or some) samples fail,
- Stress levels are reached, which are well above those expected in service, or
- Irrelevant failures occur.

An important benefit of RET is to survey and to determine the product upper and lower destruction limits. This is useful in determining the robustness of the product design by controlling them far enough from these limits.

II.3.3. Accelerated Life Tests (ALT)

Accelerated life testing consists of a variety of methods for shortening the life of products or fastening the degradation of their performance. The aim of such testing is to quickly obtain data which, properly modelled and analyzed, yield desired information on product life or performance under normal use; such testing saves much time and money [Nel-90]. The fundamental principle of accelerated testing is based on the fact that the unit under test will exhibit the same behaviour in a short time at high stresses that it will exhibit in a longer time at lower stresses [Con-01]. ALT subjects the tested units to higher-than-use stress levels to shorten their times to failure. The life data obtained are extrapolated using a life-stress relationship to estimate the life distribution at use condition. Because they yield failure information in a short time, ALTs are extensively used in various phases of product life cycle.

Early in the product design phase, the reliability of materials and components can be assessed and qualified by testing them at higher stress levels. As the design moves forward, robust reliability design is often performed to improve the reliability by choosing the optimal settings of design parameters. As soon as the design is completed, prototypes are subjected to design validation testing (DVT), which are intended to demonstrate the achievement of a specified reliability target. This type of testing often includes ALTs.

ALT can be either **qualitative** or **quantitative**, depending on the purpose of the test. A qualitative test is usually designed and conducted to generate failures as quickly as possible in the design and development phase. Subsequent failure analyses and corrective actions lead to the improvement of reliability. This type of test is known as highly accelerated life testing (HALT). Quantitative tests are aimed at estimating the product life distribution; in particular, the percentiles and the probability of failure.

Accelerated life tests are conducted on components, materials, and manufacturing processes to determine their useful life in the required product application. Their purpose is not to expose defects, but to ***identify and quantify the failures and failure mechanisms which cause products to wear out at the end of their useful life.*** For this reason, accelerated life tests must last long enough to cause the samples under test to fail by wear out. The test time may typically vary from few weeks to few months.

In practice, separate accelerated life tests are conducted for each potential wear out mechanism, since the stresses which produce failures are different for each mechanism. Traditional accelerated life test methods have involved the application of single stresses (for example, only sine vibration or only temperature cycle). However, it is increasingly felt that many potential failure mechanisms result from, or are accelerated by the environmental conditions (e.g. random vibration combined with high temperature).

Accelerated life tests are commonly called "qualification tests", because they are used to qualify components, materials or processes for given specific applications. Accelerated life tests usually take too long time to be conducted on-line, as part of any product development cycle. Therefore, they must be conducted off-line, well before the components, materials, or processes are needed for a given application. For these reasons, ALT are usually conducted generically, using generic samples which represent the materials, components and processes used for a variety of products.

The benefits of ALT are:

- The ability to estimate the useful life of the product.
- The capacity to give decision-making information for designer/manufacturer in order to identify, improve and control the critical components, materials and processes, so that the final product becomes robust and mature.

However, potential failure mechanisms must be known; and the stress environment of the product must be understood. Specific acceleration models must be available for each failure mechanism and the results must be properly interpreted.

To conclude, there is an immense impact of reliability tests on the product life-span, in terms of failure rate reduction; this impact is depicted in (figure II.4).

The gain of testing program is to obtain a more reliable product with lower hazard level during the life cycle. In wear-in and wear-out periods, this duty lies on ESS and ALT respectively, whereas RET has a comprehensive role assisting the other two types of testing.

II.3.3.1 Acceleration Methods

The purpose of acceleration is to give the reliability information more quickly. Any means that serves this purpose is an acceleration method. Basically, there are two types of acceleration methods depending on the relationship between stress and time:

A) Time independent stress application, at constant level (figure 2.4 a).

B) Time dependent stress application, this can be done in different ways, by applying different constant levels or increasing usage rate or with cyclic or random stresses; these ways are illustrated in figure II.5 (b-d) respectively. The appropriate method to use for a specific product depends on the purpose of the test and the product itself. In practice, an ALT often utilizes one or two of the four types of methods.

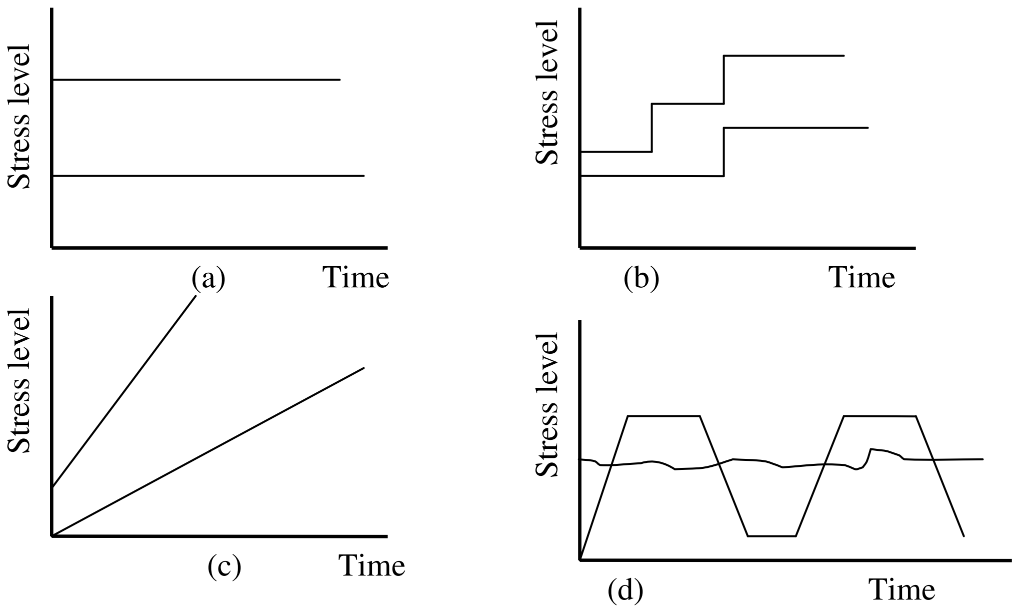


Figure II. 5 Types of acceleration methods [Yan-07]

II.3.3.2 Acceleration models

Statisticians, mathematicians and engineers have developed life-stress relationship models that allow the analyst to extrapolate a use level probability density function PDF from life data obtained at increased stress levels. These models describe the path of a life characteristic of the distribution from one stress level to another. The life characteristic can be any life measure, such as the **mean or median B(X) Life** (i.e. *the estimated time when the probability of failure will reach a specified point X % at a given stress level),* expressed as a function of stress. For example, for the **Weibull distribution**, the scale parameter η, is considered to be stress-dependent and the life-stress model for data that fits the Weibull distribution is assigned to η.

We must choose a life-stress relationship that fits the type of data being analyzed. Available life-stress relationships include Arrhenius, Eyring and inverse power law models. These models are designed to analyze data with one stress type (*e.g.* temperature, humidity or voltage). The temperature-humidity and temperature-non-thermal parameter relationships are combination models that allow us to analyze data with two stress types (*e.g.* temperature and voltage or temperature and humidity). The most important acceleration models are given in table II.1.

Table II. 1 Acceleration models [Cha-06]

Model	Formulation	descript-ion	Examples of application
Arrhenius	$t = A_0 e^{-\frac{E_a}{KT}}$ where: t: quantifiable life measure, such as mean life, characteristic life, median life and $B(x)$ life. A_0 : constant E_a : activation energy. K : Boltzmann constant ($8.62\text{x}10^{-5}$ eV/K) T: temperature (in Kelvin)	Life duration is function of thermal and chemical aging.	Electrical insulations Dielectrics Semi- conductors Inter-metallic diffusion Battery cells Lubricants and greases Plastics Lamp Filament
Inverse power	$L(V) = \frac{1}{K.V^n}$ **where** L: quantifiable life measure, such as mean life, characteristic life, median life and $B(x)$ life. V: stress level. K: model parameter to be determined, ($K > 0$). n: model parameter to be determined.	Life duration is function of non-thermal aging parameter	Electrical insulations Dielectric Bearings Lamp Filament Flash
Miner	$D = \sum_{i=1}^{k} \frac{n_i}{N_i}$ D: cumulative damage. n_i : number of applied stress cycles S_i N_i : number of stress cycles S_i at failure.	Linear cumulative damage due to fatigue	Material fatigue (at high number of cycles)
Manson-Coffin	$t = \frac{A}{(\Delta T)^B}$ t: quantifiable life measure, such as mean life, characteristic life, median life and $B(x)$ life. A and B : scale factors ΔT : Variation of temperature.	Non-Linear cumulative fatigue damage due to some cycles and /or thermal chocks	Thermal fatigue materials (exhaust-pipe), welded joints, Connection

Model	Formulation	descript-ion	Examples of application
Peck	$t = A_0(RH)^{-2.7} e^{\frac{0.79}{KT}}$ **Where** *t*: quantifiable life measure, such as mean life, characteristic life, median life and *B*(*x*) life. A_0 : Model parameter to be determined. *RH*: relative humidity. *K*: constant of Boltzmann (8.62x10^{-5} eV/K) *T*: temperature (in Kelvin)	Life duration as a function of temperature and humidity	Composite materials (epoxy)
Power of Peck	$t = A_0(RH)^{-N} f(V) e^{\frac{E_a}{KT}}$ t: quantifiable life measure, such as mean life, characteristic life, median life, *B*(*x*) life, etc. A_0 : model parameter to be determined. *RH*: relative humidity. *K*: constant of Boltzmann (8.62x10^{-5} eV/K) *T*: temperature (in Kelvin) *N*: coefficient (approximately 2.7) $f(V)$: Function of applied tension.	Life duration as a function of temperature , humidity and the tension	Corrosion
Eyring	$t = B(I_{sub}) e^{\frac{E_a}{KT}}$ ***t***: quantifiable life measure, such as mean life, characteristic life, median life, *B*(*x*) life, etc. ***B***: Model parameter to be determined I_{sub}: Maximum intensity of current in the substratum during the stress. E_a : Activation energy(from 0.1 eV to -0.2 eV). *K*: constant of Boltzmann (8.62x10^{-5} eV/K) *T:* temperature (in Kelvin)	Life duration as a function of current , in the electrical field and temperature	Hot transporters Inversion of surface Mechanical resistance

II.4 Reliability Data

One of the most critical requirements in reliability work is to know early in the life of product (preferably in the design stage) how that product will perform at given time in the future. This means that reliability information must be obtained in short time, and that it must be predictive. Most product managers have available data from prior performance of similar products, or from earlier tests, from component suppliers, or from other sources. These are usually the least expensive data available, and should be used as extensively as possible.

Table II. 2 Data-Base [Dhi-88]

N°.	Author	Title or type of data	Published in or developed in
1	M.J.Rossi	NPRD Non-electronic part reliability data, Rept.NPRD-3,1985.	(RADC) Reliability analysis center, Rome Air Development center , Griffiss Air Force Base, NY 13441-5700
2	R.E.Schafer, J.E.Angus, J.M.Finkelstein, M.Yersasi, D.W.Fulton	RADC Non-Electronic Reliability Notebook, Rept. RADC-TR-85-194, 1985	(RADC) Reliability analysis center, Rome Air Development center , Griffiss Air Force Base, NY 13441-5700
3	A.E.Green	Safety system reliability, 1983	John Wiley &Sons Chichester, UK
4	–	SYREL Reliability data bank	System Reliability Service, Safety and Reliability Directorate UKAEA, Wigshaw Lane, Culcheth, Warrington, Lancashire, WA3 4NE, England
5	–	IEEE Nuclear Reliability data Manual, IEEE Std.500-1977	Institute of electrical and electronics engineers (available from John Wiley & Sons, 605 Third Ave., New York, NY 10017)
6	–	OREDA Offshore reliability data	SINTEF Industrial Management Norway

There are many established sources (data banks) from which various types of failure data can be obtained; some of these are presented in table (II.2).

In many cases, these sources are not sufficient and experimental data must be collected. Design of experiments is a mean of obtaining quick, efficient and accurate experimental data. Design of experiments in combination with accelerated testing, can facilitate reliability prediction in relatively short time. Properly understood and applied, accelerated testing can add much value to product design.

Accelerated testing is a controversial matter, although it is widely used in almost all industries in some form or another, there are many differences of opinion about how to set up accelerated tests and interpret data collected from them. Misunderstood and improperly applied data can lead to serious mistakes. As a normal result of any test we obtain data, these data can be categorized in different types depending on the target of the testing, such as product characteristics of the life data, or some other measures of performance, such as tensile, strength or ductility.

II.5. Types of testing data

The proper analysis of life data depends on the type of data. In general, we can divide the type of testing data into two types: complete and censored (figure II.6). Censored data has basically two types according to the type of censoring, either type I (time truncation) or type II (failure truncation). Complete data consist of the exact life (failure age) of each sample unit. Figure II.7 (a) depicts a complete sample from a single test condition. In this figure the length of each line corresponds to the life-time of the corresponding test.

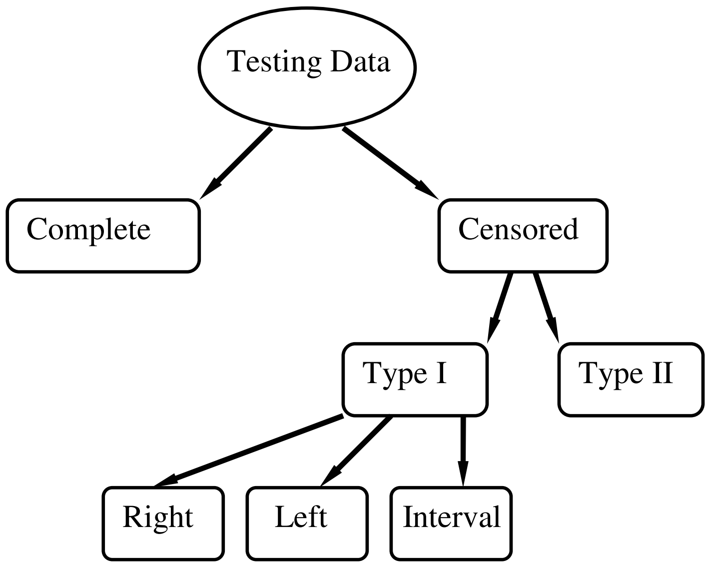


Figure II. 6 Testing data types

In practical engineering, most of life data are incomplete. That is the exact failure times of some units are unknown, and there is only partial information on their failure times. Such kind of data is called **Censored.**

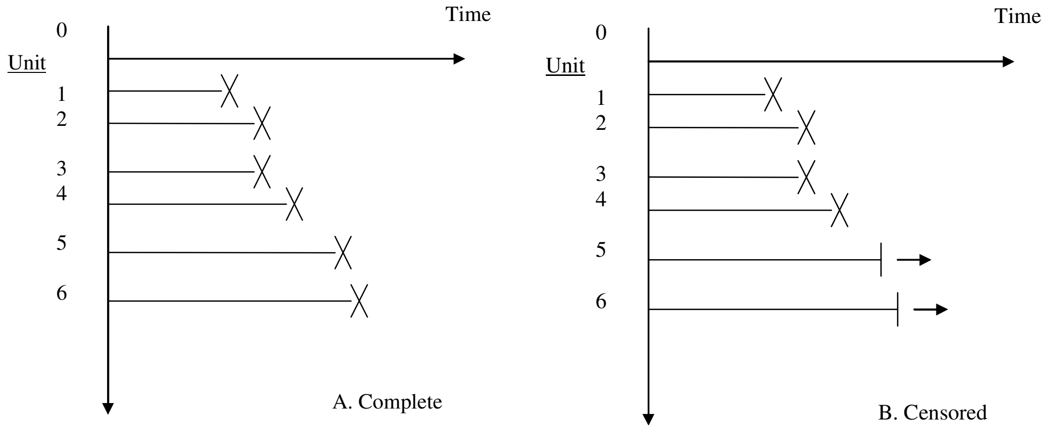


Figure II. 7 Types of life data

a. Complete data type (failure time $\times$) b. Censored data type (Failure time $\times$, running time $| \rightarrow$)

II.5.1 Censored Data

In the industrial field, we are interested in the event of time-to-failure (TTF) at the level of components and then at the level of systems. For that purpose, predicting the time-to-failure is desired through experimental approaches using statistical theory package to extrapolate the data resulting from the aimed event. Certainly, dealing with experiments comes up with data which can be classified in different categories depending on different criteria upon which we decide to stop the experiment. The experimental observation period is defined as the time elapsed since the experiment begins (time zero) until it is terminated (time T_0). However, it often occurs that we need to stop our experiment before all the elements in the sample reach the "event of interest" (e.g., failure or death). In such cases, we say that the experiment has been "suspended", "censored" or "truncated". Truncation may not be the most efficient way to conduct an experiment, from the theoretical point of view. But, due to little available time, economic or practical considerations, it happens so frequently that statistics had to find ways to deal with it in a successful manner. Some of these statistical procedures are overviewed hereafter.

II.5.1.1 Type of censoring

As mentioned above there are two types of censoring:

- **Time-Censored tests (Type I)**
  We put an end to the tests, at a pre-specified time T_0, which is independent of the event of interest (death or failure). At that time, we stop monitoring all the components as illustrated in figure II.8.

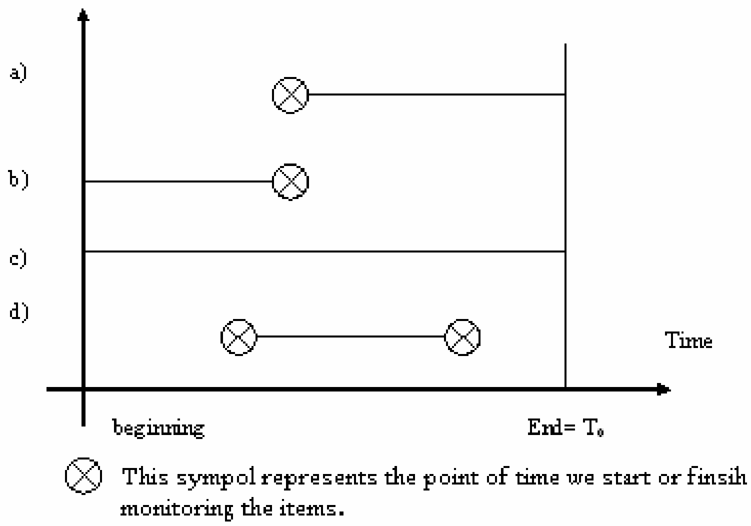


Figure II. 8 Type I (Time-Truncated) Censoring Cases

Depending on the state of the observed **component**, both at the time we start and finish our observation, we recognise three kinds:
Left–censoring: that is, when we do not know exactly at what time the life of the component started, as illustrated in case a in figureII.8. This happens because the component has already started operating before the time we begin our observation.
Right–censoring: in this case, the life may be not yet finished by the time we stop our observation. This happens when we observe the component for some time, after which we are not able to monitor it any more. This other type of truncation is known as "right censoring", case b in figure II.8.
Interval censoring: in this kind of censoring, both the beginning and end of the component "life" are unknown like the case d in figure II.8. All what we know is the time of starting and finishing the monitoring.

- **Failure-Censored tests (Type II)**

We may also chose to observe a sample of *"n"* components until the time of occurrence of some pre-specified event of interest, such as the time of the i^{th} failure $(i \leq n)$ denoted by T_i in figure II.9. Suppose that the failure times for n components are observed: $0 < T_1 < .. < T_i < \infty$. At the time of the i^{th} failure, we stop our observation of the $(n-i)$ remaining components in the operation. This censoring scheme is often referred to as "failure" or "event" truncation and is known as Type II censoring. In these cases, the stopping time T_i is random and the number of failures i occurred during experimentation is pre-established.

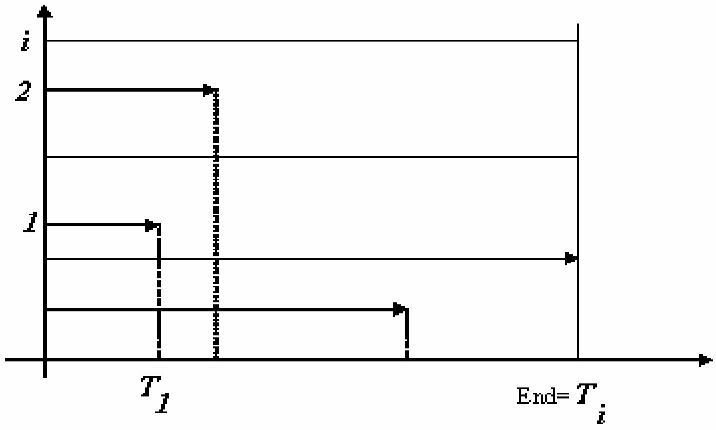


Figure II. 9 Type II (Event-Driven) Censoring Case

II.5.2 Empirical estimates of *F(t)* for non-censored data

Given a set of ordered observations (i.e., failure times) with $t_1 \leq \ldots \leq t_n$, the challenge is to identify a suitable distribution model and then to estimate its parameters. The models for estimating the cumulative distribution function CDF, $F(t)$, are used and referred in different names as:

- Empirical
- Non-parametric
- Distribution- free

These models concern only the order of the observation, not the actual value of the observation. Accordingly, these estimates are called as order statistics, the empirical estimates of $F(t)$ denoted by $\hat{F}(t)$ as rank estimators. The rank estimators are used to generate probability plots of the data, for the purpose of assessing the fit of data to a per-described distribution. Naïve estimator and mean and median rank estimators are used and depicted in table II.3, where, i is the rank of the observation $i=1, 2\ldots, n$; $F_{2(n+1-i),2i,0.5}$ is Fisher *F*-distribution.

Table II. 3 Popular Rank Estimators of F(t)

Estimator	Formula
Uniform "naïve" estimator	i/n
Mean rank estimator	$i/(n+1)$ (Herd-Johnson)
Median rank estimator	a. $\dfrac{i}{i+(n+1-i)F_{2(n+1-i),2i,0.5}}$ (exact expression) b. $\dfrac{i-0.3}{n+0.4}$ (Bernard's 1953 approximation) c. $\dfrac{i-3/8}{n+1/4}$ (Blom's 1958 approximation)

II.6. Validation testing

According to *ISO 9001:2000 Element 7.3.5(Design and development verification)* ***verification*** *is characterized as those activities involved in the evaluation of whether design outputs are properly translated from design inputs (e.g., design review, CAE, simulation), while* ***validation*** *is the term used for ongoing test activities dedicated towards demonstrating the achievement of design objectives.*

In fact, as soon as the design comes to the light, the next task is to verify whether the design has met its planned reliability targets or not. If during testing, a design fails to demonstrate the required reliability, it must be revised. Now, prior to validation testing, several tasks must be accomplished in this phase of product life cycle such as developing a test plan that specifies the test conditions (test stress and time), sample sizes, acceptance criteria and test operation procedures.

Generally, there are various difficulties and challenges in this part, such as the limitation of the sample size for testing. Although, it should be large enough from statistical point of view, the lack of time and the cost of tests lead to practical limitations. Nowadays, competitive marketplace, product designers are under immense pressure to reduced product lead times, (Lead time is the period between the initiation of any process of production and the completion of that process).

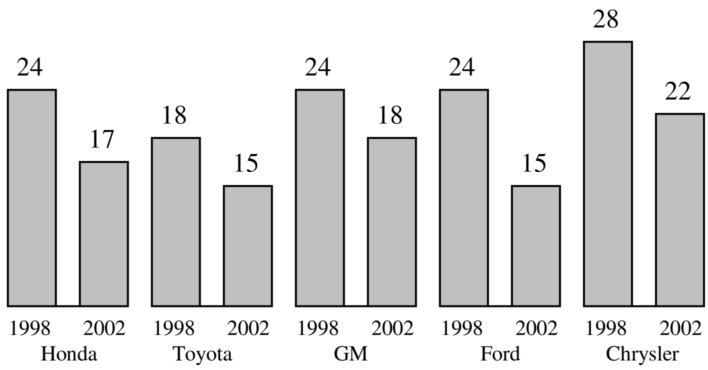


Figure II. 10 Product lead time in automotive industry [Was- 02]

In turn, test organizations have to adopt new methods for reducing design verification test time. Reliability demonstration of very complex systems can be quite costly. Companies are often unwilling to spare more than two or three units for testing. Prototypes of complex products can be extremely expensive to build. For example in 2002, the cost to build cockpit is close to $350 000 per prototype. Even subsystems as simple as window glass unit can run $30-$40,000 per prototype. Electronic circuit boards can run $5-15000 per prototype.

Here, the following question arises: Can reliability be demonstrated with a specified confidence level when sample size is too small?

This problem is an optimisation problem, because the manufacturer must trade of the cost of reliability demonstration and the number of tests or products which are feasible to be tested.

In this work, we have distinguished between two types of validation tests in principal figureII.11:

- Predetermined sample size approaches: Parametric and non parametric Bogey tests.
- Non-determined sample size approaches: parametric and non parametric sequential tests and stress resistance approaches.

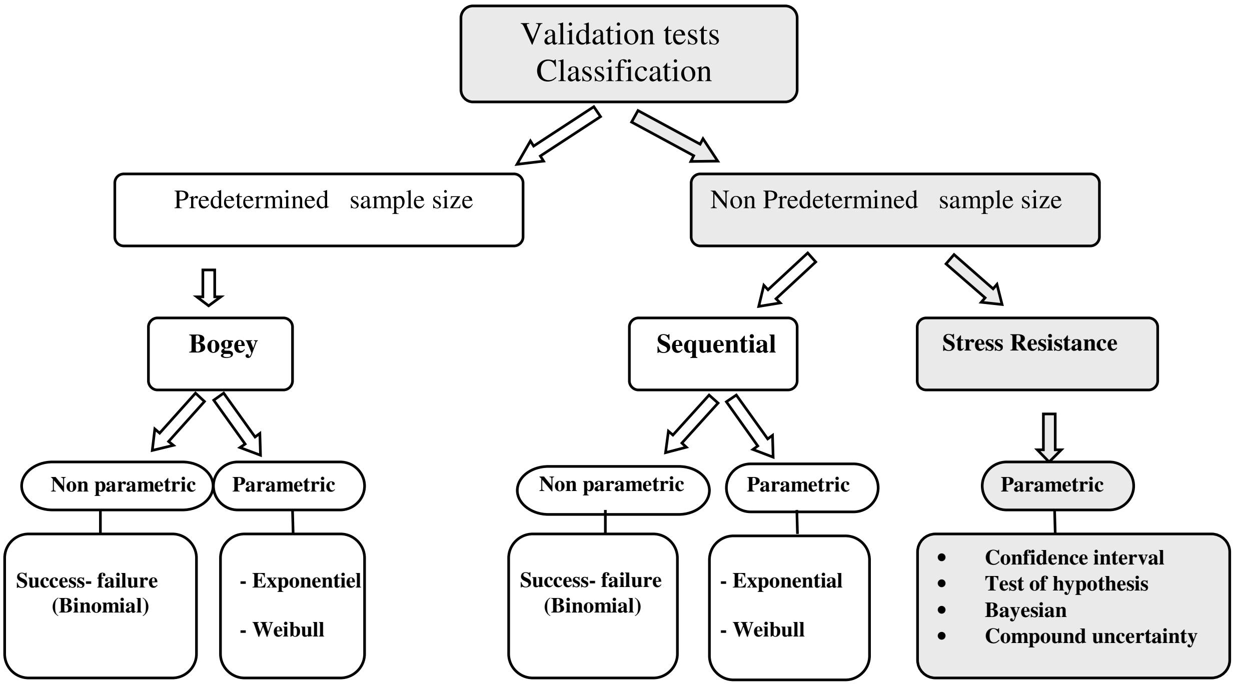


Figure II. 11 General classification of validation tests

II.6.1 Predetermined Sample size approaches

This type of tests can be classified between two categories:

II.6.1.1 Non parametric

This kind of approaches aims at determining the sample size required to meet the reliability target with a specified confidence level under Binomial assumption [Clo-34, Was-02]. This is often summarized by using $\Re$ by C notation, where $\Re$ is the reliability target and C is the prescribed confidence level. For example, an $\Re$ *95C90* reliability specification of automotive component would signify that "the likelihood or confidence that there is a 95% chance, or greater, that the component will be able to withstand the number of cycles of use *n* without incidence of sever failure is at least 90%". Mathematically, this can be expressed as: $P(\Re(n) \geq 0.95) \geq 0.90$.

- **Success–Failure test (Bogey test)**

A Bogey test [Clo-34, Was-02, Yan-07] is a test in which a fixed number of samples are run simultaneously for a predetermined time span under specified test environments. If no failure occurs, we conclude that the required reliability is achieved at the given confidence level. A Bogey test is simply characterized by the *sample size, test time and test stresses.*

Suppose that we want to demonstrate the reliability $\Re_L$ at *C%* confidence level. The minimum sample size is given by

$$n = \frac{\ln(1-C)}{\ln \Re_L} - 1 \qquad \text{(II.6)}$$

When $\Re_L$ >90%, the application of equation (2.6) leads to non realistic numbers, table II.4 illustrates the application of equation (II.6) at different reliability levels.

R_L	*Confidence level C*	*Required number of tests*
90%	80%	14
95%	80%	30
99%	80%	159

Table II. 4 Application of bogey test

Figure II.12 plots the minimum sample sizes for various values of C and $\Re_L$. It is shown that the sample size increases with the required reliability given a confidence level, or with the confidence level given a required reliability. It increases sharply when the required reliability approaches 1.

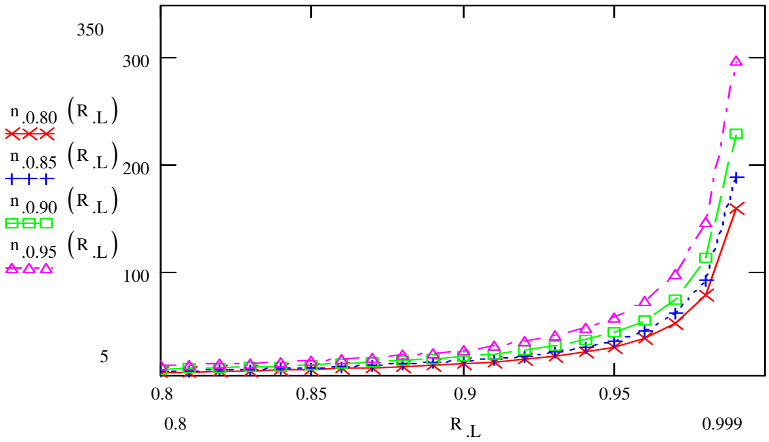


Figure II. 12sample sizes n_C at different values of C and $\Re_L$ in bogey testing

II.6.1.2 Parametric approaches under exponential or Weibull distribution

- ***Bogey and extended Bogey exponential, Weibull testing***

One can get the following results for exponential distribution:

$$P[\Re_L \le \Re] \ge C \quad \text{with} \quad \Re_L = \exp(-t_b / \theta_L) = \exp\left(\frac{-t_b \chi^2_{2r+2,1-C}}{2T}\right) \tag{II.7}$$

Equation (II.7) can be used to determine the total unit time on test requirements for any reliability requirement under exponential test planning assumptions. Taking the logarithm of both sides of equation (II.7) and rearranging the terms, we get:

$$T = -\frac{t_b \chi^2_{2r+2,1-C}}{2\ln \Re_L} \tag{II.8}$$

Now, for $r \ll n$ failures, $T \approx n.\ t_b$ and the sample size requirements for success-failure test that allows for r failures is given by:

$$n = -\frac{\chi^2_{2r+2,1-C}}{2\ln \Re_L} \tag{II.9}$$

It is possible to trade off between extending the test time t_e and decreasing the number of items required for test by a ratio called Bogey ratio:

$$m = \frac{t_e}{t_b} \tag{II.10}$$

Here, equation (II.8) is still applicable with the changes of t_b to t_e and $\Re_L$ to $\Re_{L,e}$, where t_e is the extended time of Bogey test, $\Re_{L,e}$ is the reliability at the end of extended test time t_e, which can be expressed by the following equation:

$$\Re_{L,e} = (\Re_L)^m,$$

Therefore, $Ln\Re_{L,e} = mLn\Re_L$ and $T \approx n\ t_e = n\ m\ t_b$ .Equation (II.9) becomes for the case of *extended testing*:

$$n = -\frac{\chi^2_{2r+2,1-C}}{2m\ln\Re_{L,e}} \tag{II.11}$$

Clearly, sample requirements are reduced $1/m$ times under extended testing. For the case of Weibull distribution, planning formulas are directly obtained from the exponential test using of the following substitutions: t^{β_W} for t, θ^{β_W} for θ and m^{β_W} for m. Therefore:

$$\Re_L = \exp\left(-\frac{t_b^{\beta_W}\chi^2_{2r+2,1-C}}{2T_W}\right)$$

and the test sample requirement:

$$n = -\frac{\chi^2_{2r+2,1-C}}{2m\ln\Re_L}, m = -\frac{\chi^2_{2r+2,1-C}}{2n\ln\Re_L} \tag{II.12}$$

By putting $\Re_{L,e} = (\Re_L)^{m^{\beta_W}}$, we need only to substitute m^{β} for m in the exponential, the

extended test formula given by: $n = -\frac{\chi^2_{2r+2,1-C}}{2m^{\beta_W}\ln\Re_L}$, $m = \left(-\frac{\chi^2_{2r+2,1-C}}{2n\ln\Re_L}\right)^{1/\beta_W}$ (II.13)

For the case of 0 failure (*r=0*), we can use $\chi^2_{2,1-C} = -\ln(1-C)$ and the extended testing equation (2.13) becomes:

$$n = -\frac{\ln(1\text{-}C)}{m^{\beta_W}\ln\Re_L},\ m = \left(-\frac{\ln(1\text{-}C)}{n\ln\Re_L}\right)^{1/\beta_W} \tag{II.14}$$

II.6.2 Non-predetermined Sample size approaches

Binomial, exponential, Weibull sequential life testing

Sequential life testing is a hypothesis testing situation in which the course of action is reassessed when new observations become available [Kap-77, Yan-07]. As soon as enough information is obtained to the decision, the test is stopped. Thus, the sample size is not fixed in advance but depends on the observations as they become available. However, the drawback in this approach lies in the procedure, as we have to wait till the end of the test to take a decision. It is therefore unpredictable in terms of the required time. The sequential sampling procedure will provide rules for making one of the three possible decisions. The decisions are:

1. To accept the nil hypothesis,
2. To reject the nil hypothesis,
3. To obtain additional information by carrying out another observation.

Theory of Sequential testing

Let us consider the nil hypothesis H_0: $\theta = \theta_0$, against the alternative hypothesis H_1: $\theta = \theta_1$. From the definitions of type I and type II errors in hypothesis testing, define $P[H_1/H_0] = \alpha$, *and* $P[H_0/H_1] = \gamma$, where $P[H_i/H_j]$ is the probability of accepting H_i when H_j is true, where θ is a parameter of the life distribution (e.g., an exponential or Weibull scale parameter) and θ_0 and θ_1 are the values specified for θ. Loosely, θ_0 represents the upper limit of reliability requirement above which the product lot should be accepted; θ_1 is the lower limit of reliability requirement below which the product lot should be rejected. The ratio $d = \dfrac{\theta_0}{\theta_1}$ is called the discrimination ratio. Let X be the random variable with the PDF given by $f(x_i, \theta)$. Suppose that a sequential life testing generates $x_1, x_2, ..., x_n$, which are n independent observations of X. The likelihood of the n observations is:

$$L(x_1, x_1, ..., x_n; \theta) = \prod_{i=1}^{n} f(x_i, \theta) \tag{II.15}$$

We define the ratio of the likelihood as:

$$LR_n = \frac{L(x_1, x_1, ..., x_n; \theta_1)}{L(x_1, x_1, ..., x_n; \theta_0)} \tag{II.16}$$

The ratio LR_n is also called the *probability ratio* because the sample likelihood is the joint PDF for the sample. Given a data set $(x_1, x_2 ..., x_n)$, the likelihood depends only on the value of θ. The maximum likelihood principle indicates that the likelihood is maximized when the value of θ takes the true value. We can admit that a value of θ closer to the true one would result in a larger value of the likelihood. Following the same reasoning, if θ_0 is closer to the true value of θ than θ_1, $L(x_1, x_2, ..., x_n, \theta_0)$ is greater than, $L(x_1, x_2, ..., x_n, \theta_1)$ and LR_n is less than 1.So, LR_n would become smaller when θ_0 approaches, and θ_1 leaves, the true value. It is reasonable to find a bound, say A, such that if $LR_n < A$, we would accept H_0. Similarly, we may also determine a bound, say B, such that if $LR_n > B$, we would reject H_0. If LR_n is between the bounds, we would fail to accept or reject H_0 thus, the test should be continued to generate more observations. The decision rules are as follows:

- Accept H_0 if $LR_n \leq A$.
- Reject H_0 if $LR_n \geq B$.
- Draw one more unit and continue the test if $A \leq LR_n \leq B$.

By following the above decision rules and the definitions of type I and type II errors, we can determine the bounds as $A = \gamma / 1 - \alpha$, $B = 1 - \gamma / \alpha$ where α is the type I error (supplier's risk) and β is the type II error (customer's risk). In many applications, it is computationally more convenient to use the log likelihood ratio, namely, $\ln(LR_n) = \sum_{i=1}^{n} \ln\left[\dfrac{f(x_1; \theta_1)}{f(x_1,; \theta_0)}\right]$ then the continuous test region becomes $\ln\left(\dfrac{\gamma}{1-\alpha}\right) < \ln(LR_n) < \ln\left(\dfrac{1\text{-}\gamma}{\alpha}\right)$.

The number of tests required for reliability demonstration in this technique depends on the position of lines between reject, continue and accept regions as shown in figure II.13.
The exact position will depend on:

- o The discrimination ratio or the error in estimating failure intensity one is willing to accept.
- o The customer level of risk or the probability one is willing to accept of falsely saying the failure intensity objective has been met when it is not.
- o The supplier level risk or the probability one is willing to accept of falsely saying the failure intensity objective has not been met when it is.

When risk levels and/or the discrimination ratio decrease, the continue region becomes larger. This situation requires more testing before reaching either accept or reject region

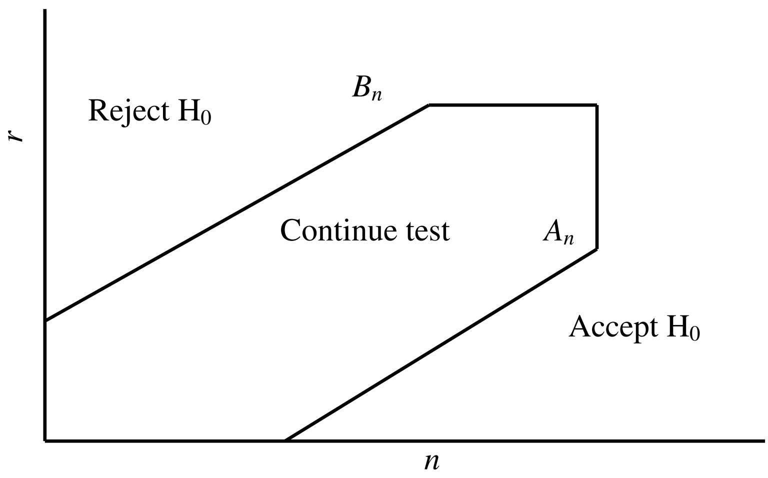


Figure II. 13 Graphical binomial sequential test plan

II.6.3 Stress resistance based approaches

In this section, we tackle the subject from another point of view in order to reflect practical situations. In fact, in many sensitive industries, such as military and aerospace, no more than one or two units can be tested. The situation is even worst if no failure has been observed within the specified time! In automotive, engine fatigue tests cannot be run on more than few units (i.e. five), as production cannot wait for the outcome of test results. In this case, the obtained information is very limited and additional uncertainties are strongly involved in the reliability model. In such situation, it is not easy to trust on the reliability demonstration tests. As the reliability of the distributed product remains practically unknown, the reliability engineer tries to state that he can guarantee the target (i.e. product resistance) with a certain confidence.

In the present work, four probabilistic approaches based on structural reliability are considered to achieve the reliability demonstration target; namely: confidence intervals, test hypothesis, Bayesian approach and compound uncertainties. The latter is proposed and developed for reliability demonstration, as robust estimation is shown for small sample sizes.

For a limited number of tests, the estimates of the mean and standard deviation of resistance are random, as they are strongly dependent on the drawn sample. In the following sections, the methods dealing with these estimates are discussed and a compound uncertainty method is proposed. It is assumed that the resistance distribution type and the coefficient of variation (COV) are already known from previous state of knowledge. The PDF of the applied load is also assumed to be known. The predefined reliability target β_T can thus be guaranteed by putting the mean objective resistance of the product m_R^{obj} sufficiently far from the mean load m_S; i.e. $m_R^{obj} = k\ m_S$, where k is a global safety factor.

The following assumptions are admitted in the present work:

- Tests are conducted either up to failure when the test load is not specified, or under a constant stress S_{test} whether failure is observed or not.
- The distribution types are defined for stress and resistance.
- The stress parameters (mean, standard deviation …) are known.
- The coefficient of variation of resistance is known (or assumed).

II.6.4.1 Confidence interval method

This approach consists in defining the confidence interval bounds for the parameter estimated from tests. The confidence interval for the mean resistance takes the form: $\overline{R}_{low} \le m_R \le \overline{R}_{up}$, where the subscripts "*low*" and "*up*" refers to lower and upper bounds, respectively. These bounds depend on the number of tests and the desired confidence level. In order to ensure the product reliability, the lower bound of the test mean must be greater than the target population mean: $\overline{R}_{low} \ge m_R^{obj}$, as illustrated in figure II.14. This condition ensures conservative bounding of the product mean resistance. Knowing the distribution type and the coefficient of variation, the specification of the lower bound of the mean resistance of the tested sample allows us to guarantee the prescribed reliability target.

The reliability demonstration procedure is illustrated in figure. II.14. To guarantee the target reliability of the product, it is necessary to ensure that the population mean is set above the objective level m_R^{obj}; i.e. if $m_R = m_R^{obj}$, then $P_f = P_f^{obj}$, where P_f^{obj} is the admissible failure probability. The tests conducted on a limited number come up with an apparent PDF for resistance R^{test}, with mean estimate $\overline{R}$ which is also a random variable with mean $m_{\overline{R}}$ and standard deviation $s_{\overline{R}}$. Contrary to what is usually applied in practice, the mean estimate is not normally distributed for small sample size, but it depends on the distribution of R, as the central limit theorem does not apply. For a given confidence level (1-α), the tests can guarantee the objective mean resistance m_R^{obj} by setting the unilateral lower bound of $\overline{R}$, denoted $\overline{R}_{low}$, as:

$$\Pr\left[\overline{R} \le \overline{R}_{low} = m_R^{obj}\right] = \alpha \tag{II.17}$$

Under the normality assumption (accepted for at least five specimens), the above expression leads to:

$$\overline{R} \geq m_R^{obj} + u_{\alpha/2} \frac{\sigma_R}{\sqrt{n-1}} \tag{II.18}$$

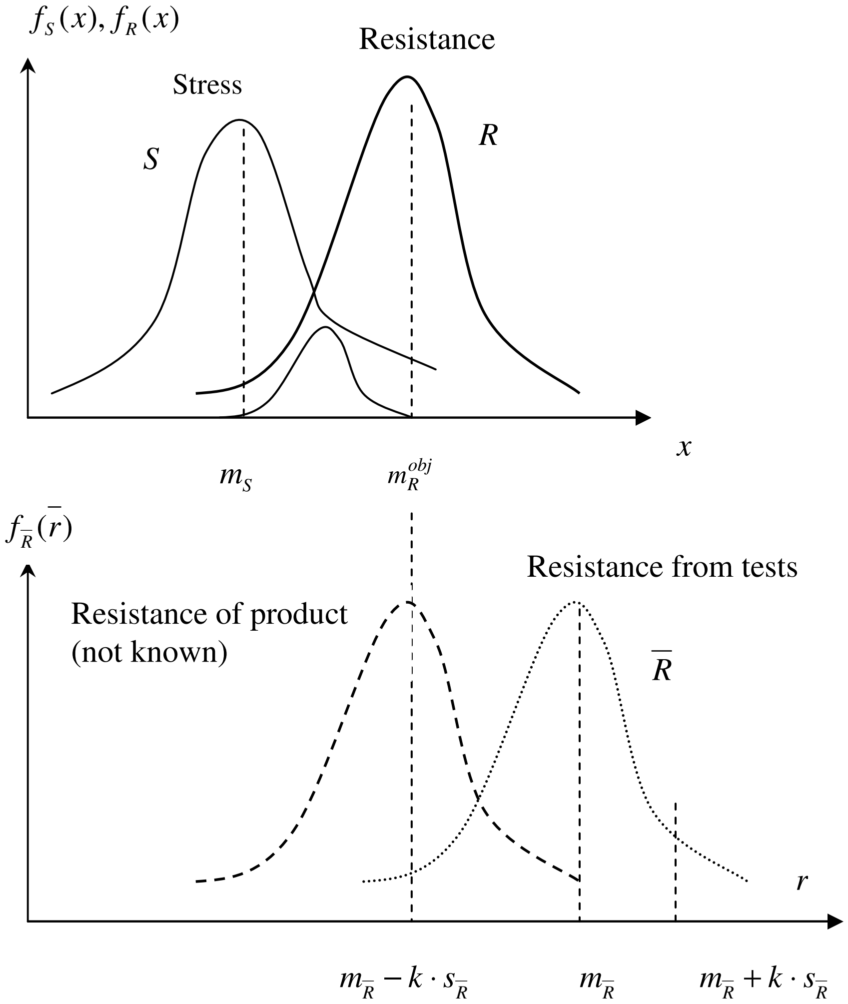


Figure II. 14 Principle of reliability demonstration.

Given that, $c_R = \frac{\sigma_R}{m_R^{obj}} \approx \frac{\sigma_R}{\overline{R}_{low}} = \frac{\sigma_R}{k \cdot m_S}$, equation (II.17) can be re-written in the form:

$$\overline{R} = \left(1 + \frac{u_{\alpha/2} c_R}{\sqrt{n-1}}\right) \cdot m_R^{obj} \tag{II.19}$$

The term between parentheses defines the test uncertainty contribution and can be called the test factor: $F_t = 1 + \frac{u_{\alpha/2} c_R}{\sqrt{n-1}}$; it depends on the confidence level, the number of tests and the resistance coefficient of variation. The procedure is thus to conduct tests up-to failure and the decision making can be formulated as following:

> "The product reliability is demonstrated if the mean resistance obtained from tests is greater or equal to the value defined by the test factor F_t".

If this statement is not met, the prescribed reliability cannot be demonstrated. This approach is commonly used in industry, because of its simple understanding as an extension to deterministic safety factor approach. However, its main disadvantage lies in the lack of robustness as it leads in many cases to largely over-designed products. In fact, as the sample size is usually low, the confidence interval is wide and it is necessary to increase the margin between $\overline{R}$ and m_R^{obj}. This leads to the aberrant design rule: to save time and money by making few tests, we have to spend a lot of money by over-designing the product!

In practice, when testing few specimens, the obtained mean test resistance may be either smaller or higher than required. The engineer cannot identify whether it is due to large scatter of test specimens or due to bad product quality. If small mean resistance is observed, new tests can be performed to try to increase the mean resistance in order to prove the product reliability (if it is the case). On the other case, if high mean resistance is observed by testing, we have no interest to conduct new tests, as lower resistances may be observed! Naturally, this procedure has a lack of consistency, as the test maker can play to demonstrate wrongly the product reliability, by tuning the test policy.

II.6.4.2 Test hypothesis approach

This approach is based on the sampling theory, such as wrong acceptation of bad products. Figure II.15 illustrates the decision possibilities at the end of the test procedures. According to the test of hypothesis theory, two types of error exist and the choice of the threshold is not trivial, since decreasing one type increases the other [Mee-98, Ben-70]. Although the test of hypothesis is mainly based on error type I, error type II considerably increases with the decrease of the number of tests. Two possible approaches can be followed, by considering either one or both errors.

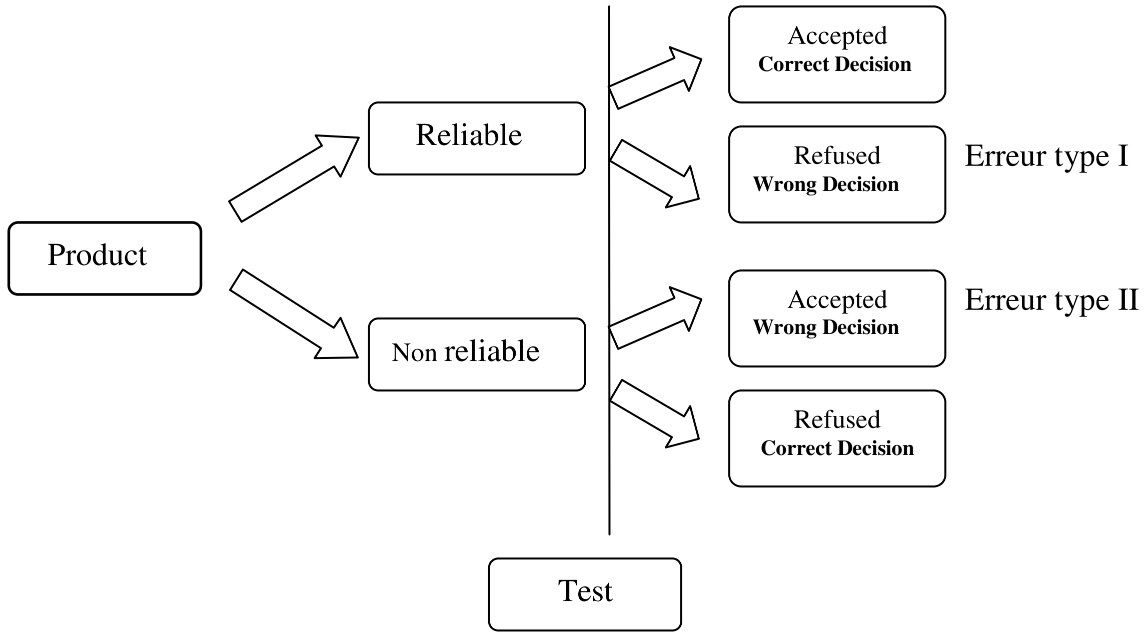


Figure II. 15 Decision making at the end of the tests.

Case 1: Decision according to one criterion

In reliability demonstration tests, the test load must be set to assure that the reliability is larger than the objective, i.e. $\Re \geq \Re_{obj}$, with a given confidence level α; where the reliability $\Re$ is determined by the tests and $\Re_{obj}$ is the objective reliability for the product. The problem of decision making is defined by two hypotheses:

H_0: $\Re < \Re_{obj}$, the objective reliability is not satisfied

H_1: $\Re \geq \Re_{obj}$, the objective reliability is satisfied

The most critical error (i.e. error type I) is to wrongly decide that the product is reliable, while it is not; i.e. wrongly reject H_0 with confidence level 1- α . For n independent tests conducted under the test load S_{test} , the probability to observe no failure when H_0 is true is:

$$\Pr_{H_0}\left[N_f = 0\right] = \Pr_{H_0}\left[R_1 > S_{test}, R_2 > S_{test}, \ldots\ldots, R_n > S_{test}\right] = \left[1 - F_R\left(S_{test} \middle| m_R = m_R^{obj}\right)\right]^n \leq \alpha \tag{II.20}$$

where N_f is the number of observed failures, R_i is the resistance of the i^{th} specimen and $F_R(\cdot)$ is the resistance CDF conditioned by $m_R = m_R^{obj}$ (i.e. reliable product). For a confidence level α, it becomes:

$$F_R\left(S_{test} \middle| m_R = m_R^{obj}\right) = 1 - \sqrt[n]{\alpha} \tag{II.21}$$

This expression allows us to set the test load in terms of the sample size and the confidence level, as follows:

$$S_{test} = F_R^{-1}\left(1 - \sqrt[n]{\alpha} \middle| m_R = m_R^{obj}\right) \tag{II.22}$$

The test factor in this approach can be defined as:

$$F_{test} = \frac{S_{test}}{k\ m_S} = \frac{F_R^{-1}\left(1 - \sqrt[n]{\alpha} \middle| m_R = k\ m_S\right)}{k\ m_S} \tag{II.23}$$

If R and S are normally distributed, equation (II.23) takes the form:

$$F_{test} = 1 + c_R \Phi^{-1}\left(1 - \sqrt[n]{\alpha}\right) \tag{II.24}$$

The demonstration procedure consists in conducting a number of tests n under the prescribed stress S_{test}, if no failure is observed, the product reliability is satisfied, otherwise the demonstration fails. It is also possible to derive similar formula for a given number of failures $N_f > 0$ among the number of tests N. In the case where the number of observed failures is less or equal to N_f, the test stress S_{test} is defined by the expression:

$$\sum_{i=0}^{N_f} C_N^i \left[F_R\left(S_{test} \middle| m_R = m_R^{obj}\right)\right]^i \times \left[1 - F_R\left(S_{test} \middle| m_R = m_R^{obj}\right)\right]^{N-i} = \alpha \tag{II.25}$$

Case 2: Decision according to two criteria

This approach considers two types of risks [Pon-05]:

- Supplier risk: refusing reliable product (error type I); this probability must be limited to α.
- Customer risk: accepting wrongly unreliable product (error type II); this probability must be limited to γ.

Under random load effect, the product can be defined in terms of acceptable and non-acceptable reliability, as illustrated in figure.II.16.

For reliable product $P_f < P_{f_{obj}}$: we write $P_{f_{acc}} = P_{f|m_R \geq m_R^{obj}}$.

For non-reliable product $P_f > P_{f_{obj}}$: we write $P_{f_{ina}} = P_{f|m_R < m_R^{obj}}$

where $P_{f_{acc}}$ and $P_{f_{ina}}$ are respectively the acceptable and non-acceptable failure probabilities, corresponding to the failure probability conditioned by sufficient and insufficient mean resistance. For a given test load S_{test} and under the assumption of constant coefficient of variation, the conditional failure probabilities are given by the following equations:
For non-reliable product: $P_1 = \Pr\left[S_{test} \geq R \middle| m_R < m_R^{obj}\right]$
For reliable product: $P_2 = \Pr\left[S_{test} \geq R \middle| m_R \geq m_R^{obj}\right]$

Figure II.16 illustrates the probability distributions of resistance for reliable and non-reliable products. Actually, the probabilities P_1 and P_2 depend on the test stress S_{test}. Equivalently, it is possible to write the probability of equipment surviving by: 1-P_1 and 1-P_2, for the cases of non-reliable and reliable respectively.

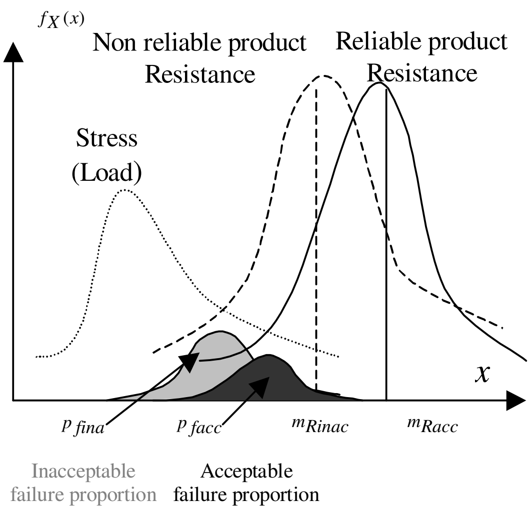


Figure II. 16 Probability distributions of reliable and unreliable products

For n tests where no failure is observed, the probability of refusing non-reliable product is calculated as follows:

$$\begin{aligned} P_{refuse|\Re<\Re_{obj}} &= \Pr\left[\text{at least one failure is observed} \middle| \Re < \Re_{obj}\right] \\ &= 1 - \Pr\left[S_{test} < R_1, S_{test} < R_2, \ldots, S_{test} < R_n \middle| \Re < \Re_{obj}\right] \\ &= 1 - (1 - P_1)^n \end{aligned} \quad \text{(II.26)}$$

Given the confidence level, this probability should be higher than γ, i.e. $P_{refuse|\Re<\Re_{obj}} \geq \gamma$.

Similar developments are performed for the second type of risk. The probability of refusing a reliable product is the probability of the failure for reliable product:

$$\begin{aligned} P_{refus/\Re>\Re_{obj}} &= \Pr\left[\text{at least one failure is observed} \middle| \Re > \Re_{obj}\right] \\ &= 1-\Pr\left[S_{test} < R_1, S_{test} < R_2, \ldots, S_{test} < R_n / \Re > \Re_{obj}\right] \\ &= 1-(1-P_2)^n \end{aligned} \tag{II.27}$$

This probability should be kept below the confidence level α; i.e. $P_{refuse|\Re<\Re_{obj}} \leq \alpha$.

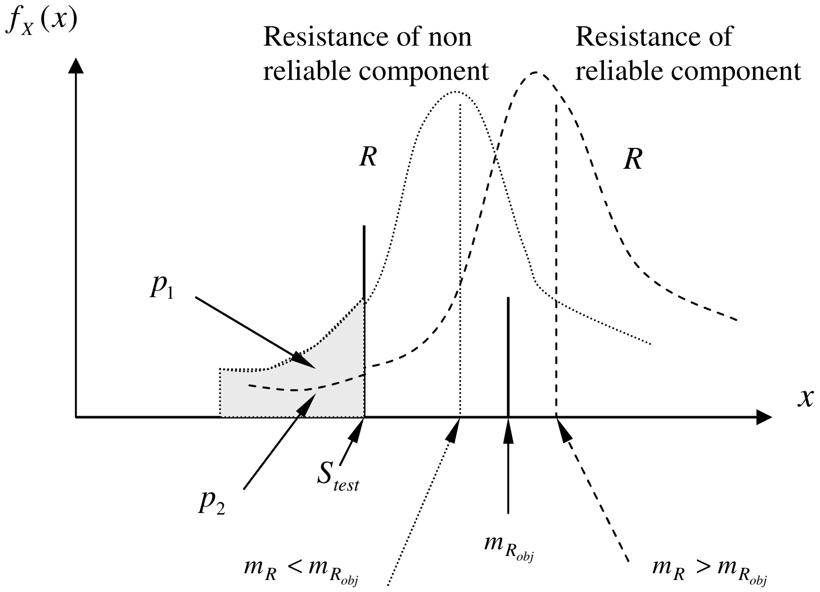


Figure II- 1 Probability of failure for reliable and non reliable product

Finally, the problem of defining the demonstration tests consists in specifying the number of tests and the stress level by solving the following two equations:

$$\begin{aligned} &1-(1-P_1(S_{test}))^n \geq \gamma \\ &1-(1-P_2(S_{test}))^n \leq \alpha \end{aligned} \tag{2.28}$$

II.6.4.3 Bayesian approach

This approach is convenient for small number of tests, each new information allows us to update the prior idea about the product resistance. When new information is obtained about a product, it must be processed to improve the prior estimation of reliability. For example, if the initial distribution of the mean resistance is $f_{m_R,0}(m_R)$, and a new test is carried out, leading to the observation r_1, the Likelihood function is written by: $L(r_1|m_R) = f_{R|m_R}(r_1|m_R)$ and the updated mean resistance distribution can be given by:

$$f_{m_R,1}(m_R|r_1) = \frac{L(r_1|m_R)\cdot f_{m_R,0}(m_R)}{\int_D L(r_1|m_R)\cdot f_{m_R,0}(m_R)\ dm_R} \tag{II.29}$$

To find the test load, we have to answer the question: what is the probability that the product fits the target reliability, given that the observed mean value of resistance from test is $m_{\overline{R}} = S_{test}$? Therefore, the test load S_{test} must be adjusted to satisfy the condition that the probability must be greater or equal to the threshold of confidence 1-α.

$$\Pr[m_R \geq m_R^{obj} | m_{\overline{R}} = S_{test}] = 1-\alpha \tag{II.30}$$

The solution can be found by iterative methods using the posterior density function. Find S_{test} which satisfies:

$$\Pr[\overline{R} \leq m_R^{obj}] = F_{\overline{R}}^{'}(m_R^{obj} | m_{\overline{R}} = S_{test}) \leq \alpha \tag{II.31}$$

The drawback in the Bayesian approach lies in the influence of the prior information, which may deviate considerably from the product data. While the Bayesian approach is very attractive, the updating process may lead in some cases to non-conservative estimates, depending on the selection of the new test specimen.

- **Choice of prior distribution [Lan-05]**

  One of the main difficulties of the Bayesian approach remains the choice of a probability distribution a priori appropriate to the state of initial data.

- **Informative-ness of prior data**

  Very often subjective knowledge is relatively vague, and it is difficult to specify prior precise statistical law to represent it.

  The properties must be associated with this law are:

  - the calculation of posterior density which lies between the prior distribution and the distribution obtained from observations which must be simple;
  - the posterior distribution must be of the same type as the prior distribution, in order to allow an iterative calculation.
  - the prior distribution must be able to represent a large number of cases;
  - it must be parametric and the parameters must be able to be interpreted physically;
  - the rules of coherence and good sense must be respected.

  Prior information available from past experience before the collection of test feedback must be updated with data more recent.

  The importance of the choice of prior probability density depends on:

  - Relative informative-ness of data with respect to tests observations feedback: this is what will impose the choice of a prior distribution appropriate to the whole data available.
  - Representation of the studied physical phenomenon, characterized by the likelihood function and conditional to the observations.

II.6.4.4 Compound Uncertainties approach

The main idea behind this approach is to consider the test sample estimates as additional random variables in the reliability analysis model, rather than just a distribution parameter. This leads to compound variable definitions (i.e. random variables whose distribution parameters are random). Having these compound variables, the probability distribution is conditioned by the parameter estimate $f_{X|m,\sigma}(x|m,\sigma)$, as illustrated in figure II.17.

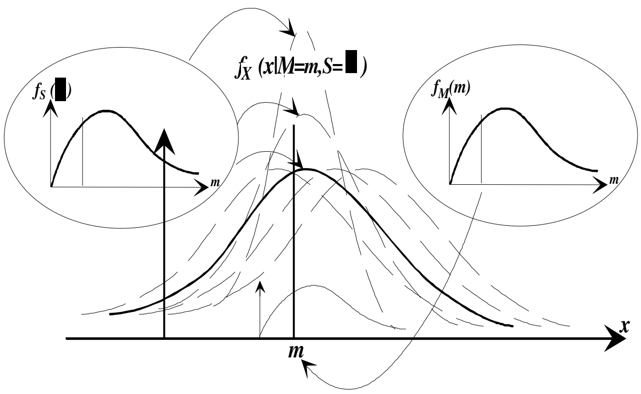


Figure II. 17 Compound distribution of the resistance

In fact, the computed reliability becomes also random, as it depends on the randomness of the sample mean and standard deviation. For a given observation of the mean resistance $\bar{r}$, the failure probability of the product in operation can be written as:

$$P_f\left(\overline{R}=\bar{r}\right)=\Pr\left[R\left(\bar{r},c_R\right)-S\leq 0\right]\leq P_{f_{obj}} \tag{II.32}$$

As the mean estimate $\overline{R}$ is also random, following the distribution $f_{\overline{R}}(\bar{r})$, the failure probability for a mission can be computed by:

$$P_f=\int_0^{\infty}P_f\left(\bar{r}\right)\cdot f_{\overline{R}}(\bar{r})d\bar{r} \tag{II.33}$$

When tests are performed at the stress level S_{test}, the product reliability, guaranteed at the confidence level (1-α), can be written as:

$$\Pr\left[\Pr\left[R\left(\overline{R},c_R\right)-S\leq 0\right]\leq P_{f_{obj}}\mid m_{\overline{R}}=S_{test}\right]=1-\alpha \tag{II.34}$$

This expression allows us to specify the test load that ensures the objective of in-service reliability, with a confidence probability of 1-α. It can be directly solved, in order to define the test load S_{test}. In the case of normal distributions of *R* and *S*, equation II.32 can be equivalently written in the form:

$$\Pr\left[\frac{\overline{R}-m_S}{\sqrt{\left(c_R\overline{R}\right)^2+\sigma_S^2}}-\Phi^{-1}\left(P_{f_{obj}}\right)\leq 0\right]=1-\alpha \tag{II.35}$$

This is solved for the test load as:

$$S_{test} = m_{\overline{R}} = \frac{m_S + \Phi^{-1}\left(P_{f_{obj}}\right)\sqrt{c_R^2 m_R^2 + \sigma_S^2}}{1 - \Phi^{-1}(1-\alpha)c_R} \qquad \text{(II.36)}$$

This approach has the advantage of considering both in-service and test uncertainties, where the coupled effect plays an important role in reliability demonstration. In other words, random strength may have large influence on product reliability, requiring high test precision on its parameters. On the opposite, when resistance does not have much influence on product reliability (due to large dispersion of operational loading for example), it becomes useless to get high precision on the resistance parameters and testing costs and time can be saved.

II.6.4.5 Numerical examples

In order to illustrate the different methods for reliability demonstration, two examples are considered, where normal and non-normal distributions are analyzed.

1. Normally distributed limit state

In this example, the stress-resistance interference model is considered: $G(R,S) = R - S$, where *R* and *S* are normally distributed random variables with the parameters indicated in table II.5.

VARIABLE PARAMETERS

Variable	Mean	Coefficient of variation
Resistance R	To be determined	0.10
Stress S	180 [MPa]	0.15

Table II. 5 Normal distribution problem parameters

To ensure the target reliability index $\beta_T = 4.0$ (corresponding to the failure probability $P_f = \Phi(-4) = 3.2\times10^{-5}$), the objective mean resistance for the product is obtained by solving:

$$\beta_T = \frac{m_R - m_S}{\sqrt{(c_R m_R)^2 + (c_S m_S)^2}} = \frac{m_R - 180}{\sqrt{(0.10\, m_R)^2 + (0.15 \times 180)^2}} = 4 \tag{II.37}$$

The solution leads to $m_R^{obj} = 360$, which becomes the design target for the distributed products. To demonstrate the reliability, five tests are sequentially performed to measure the product resistance.

Table II.6 shows the observed resistances during these tests (*r_1=433.1, r2=326.9…r_5=326.4*) for any number of tests (from 1 to 5), the mean resistances are given in the second line of the table; i.e. the first value in line 2 is the mean value of first test, then the second value in line 2 is the mean value of the first two tests (i.e.(433.1+369.9)/2=398 MPa) and so on. The following lines in the table give the resistance values resulting from applying each one of the above approaches. For these approaches, the reliability demonstration decisions are indicated:

Test number	1	2	3	4	5
Resistance*	*433.1*	*362.9*	*393.7*	*396.4*	*326.4*
Mean **	*433.1*	*398.0*	*396.6*	*396.5*	*382.5*
Conf. Interval	419.2	401.8	394.2	389.6	386.5
Demonsration**	OK	NO	OK	OK	NO
Test of Hyp. 1	504.3	390.5	407.0	399.1	322.3
Demonstration	NO	NO	NO	NO	OK
Test of Hyp. 2	406.1	377.2	363.2	354.3	348
Demonstration*	OK	NO	OK	OK	NO
Bayesian	299.4	325.9	336.1	343.5	334.1
Demonstration***	NO	NO	NO	NO	NO
Compound	422.7	339.2	338.6	332.6	322.9
Demonstration**	OK	OK	OK	OK	OK

Objective Value 360***

Table II. 6 Test results and demonstration methods

For the confidence interval approach, the lower bounds at the confidence level of **0.05** shows that the test load decreases from **419.2** to **386.5**, when the number of tests goes from one to five; the values obtained from tests 1, 3, and 4 are higher than the observed mean resistances and therefore reliability is not demonstrated in theses tests.

When applying the test of hypothesis, the probability of accepting the non-reliable product is set to 0.05 (i.e. γ=0.95) and the probability of refusing reliable one is set to α=0.05. In case of controlling only one hypothesis, the test load for the first four tests cannot satisfy the zero failure condition. Because at least one of the observed resistances is lower than the test load. In this case, five tests are required to demonstrate the reliability. In fact, only the test load of 322.3 is lower than all the five specimen resistances and therefore, reliability is only demonstrated for this value. Thus, according to this approach, the reliability is not demonstrated and the product is rejected.

When applying the two-criteria on approach, the test loads are lower than those for the case of one criterion. Only the case of the first four tests allows us to demonstrate the product reliability. When carrying out the fifth test, very low resistance is observed (r_5=326.4) and the approach fails again to justify the reliability at the load level of 348 MPa.

The Bayesian results represent the lower bounds for product resistance, updated by the tested specimens. The reliability is demonstrated when these lower bounds are higher than the objective mean resistance (360 MPa). Clearly, in table II.6, this approach fails for all the five tests.

For the purpose of testing the robustness of Bayesian approach, the prior distribution is considered normal with mean value equal to 250 MPa and coefficient of variation equal to 0.15. This choice is pessimistic and far from the targeted value lead to verify the speed of convergence of this approach.

Finally, the compound uncertainty method gives the mean test load that should be confirmed by experiments. It can be shown that this load is lower than the observed mean resistance for all the tests, and therefore reliability is demonstrated.

2. Bearing with Weibull lifetime distribution

In this example, the reliability of mechanical bearing is considered, where the time-to-failure is given by a Weibull distribution (figure II.18), where the parameters are drawn from technical notes:

$$F_{T_f}(t) = 1 - e^{-\left(\frac{t-0.02}{\eta}\right)^{1.483}} \qquad \text{(II.38)}$$

where t is expressed in thousands of hours. The scale parameter η characterizes the life span of the bearing.

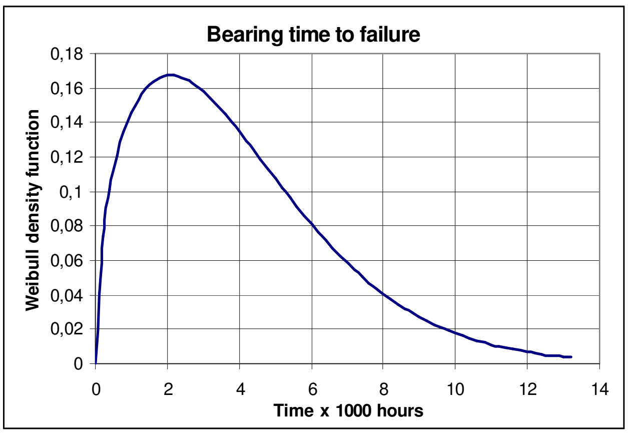


Figure II. 18 Lifetime distribution of bearing.

The limit state function is given by the difference between the time-to-failure and the operation service time:

$$G(T_f\,, T_{operation}) = T_f - T_{operation} \tag{II.39}$$

The operation time $T_{operation}$ is normally distributed with mean of 140 hours and coefficient of variation equal to 15%. The admissible failure probability for this problem is set to $4.7\text{x}10^{-3}$. For this level, the scale parameter of the bearing production should be equal to 4439 hours. It is to be noted that, for these parameters, the standard deviation is equal to 2753h. The objective failure time has been computed from structural reliability theory, its value is equal to 4403 hours.
It is now required to justify by demonstration tests that the safety level can be ensured. For this purpose, five tests have been carried out and the observed times to failure are indicated in table II.7.

Test number	1	2	3	4	5
*Failure time**	*4321*	*3741*	*2524*	*5316*	*2853*
*Mean***	*4321*	*4031*	*3529*	*3975*	*3751*
Conf. Interval	2616	2826	2545	3123	2989
Demonstration	OK	OK	OK	OK	OK
Test of Hyp. 1	9325	5835	4450	3675	3175
Demonstration	NO	NO	NO	OK	NO
Test of Hyp.2	8190	5141	3915	3228	2781
Demonstration	NO	NO	NO	OK	OK
Bayesian	3638	3632	3602	3639	3612
Demonstration	NO	NO	NO	NO	NO
Compound	1900	1580	960	3100	1131
Demonstration	OK	OK	OK	OK	OK

Objective Value 4403***

Table II. 7 Bearing problem results

The results of applying the different approaches have been arranged in the same order as in the previous example. This time, the confidence interval method has demonstrated the reliability; whereas, one- and two-criterion test hypothesis as well as Bayesian approaches, fail again to demonstrated reliability. The compound uncertainty has been successfully demonstrated reliability in this example also.

For both examples, the compound uncertainty method has shown to be robust, as it gives stable characterization of the product reliability. It allows us to consider the coupled effect of test and in-service uncertainties, avoiding therefore excessively cumulated safety margins.

Discussion

The comparison between the different methods for reliability demonstration with little number of tests shows that the confidence interval approach shows largely perturbed results. It gives over-estimated values in the first example, whereas, it has been able to demonstrate reliability in the second example with a high safety margin. Clearly, the lack of robustness represents its main disadvantage. The test of hypothesis based on one criterion exaggerates the test load required for reliability demonstration, especially for the first three tests. The two criteria approach allows us to reduce the test load depending on the acceptable percentage value, although some values have got demonstrated reliability. The Bayesian approach is still unable to reach the target of reliability demonstration, as it failed to define robust and stable rule for product qualification. Finally, the compound uncertainties approach has succeeded in this task, by taking into account all uncertainties related to stress and resistance. It has demonstrated the robustness in the two examples, without exaggerating test severity. So, in our opinion it is advisable to use it in reliability demonstration task.

II.7 Design and validation cost optimization

Life cycle cost analysis is a tool for choosing the most cost-effective approaches from a series of alternatives. It is mainly committed by early design stages. In this part of the work, we combine the design cost and the validation test cost in order to define the minimum cost with minimum number of tests. This answers the question what is the optimal number of tests which satisfies the reliability target and gives the demonstration necessary to validate the product under certain confidence level? Certainly, lot of factors can be maneuvered such as the number of products, the cost of product, the cost of tests, the total number of tests, etc. This kind of problems is considered as a decision making problem according to the hierarchy of structural reliability measures [Mel-99].

General expression for life cycle cost is introduced by **[Kle-04]** as follows:

LCC= design cost + validation cost+ manufacturing cost +warranty cost + overhead (II.40)

where LCC is life cycle cost. For our case, we have tackled the problem taking into account the first tow terms. Therefore, for our case, equation (II.40) becomes.

LCC= design cost (initial cost) + validation cost (II.41)

Equation (II.41) represents the two major quantifiable characteristics in products life cycle: the reliability and quality.

The reliability is implied in the initial cost as we set a certain level of reliability as a target to be attained under certain level of confidence. Studies reported in [Dow-92] that the design of product influences between 70 % and 85% of the total cost of a product. Therefore, designers can substantially reduce the LCC of products by giving sufficient consideration to the design.

The quality is represented by reliability demonstration or validation (second term), i.e. the product is conformed to design specification. Recalling that compound uncertainties has demonstrated the most rational approaches, compared to the other methods, we have chosen this approach in our model to estimate mean resistance of the design problem.

Figure II.19 provides the general outline of the LCC model and consists of two curves. The first curve is the descending one which represents the design cost with number of tests

$$C_{design} = n_p m_R \gamma_P \tag{II.41}$$

where n_P is the total number of products, m_R is the mean resistance defined to guarantee the target reliability, and γ_P is the cost of unit product.

The second is the ascending curve represents the validation or testing cost C_{test} which can be expressed mathematically by:

$$C_{test} = n_{test} \gamma_{test} \tag{II.42}$$

where n_{test} is the number of tests required for reliability demonstration, γ_{test} is the cost of performing one test. Validation activities are defined as the formal process of confirming through testing analysis, inspections and other engineering activities that product reliability is met, the cost of these activities includes energy, labour, maintenance, depreciation of equipments and miscellaneous [Kel-07]. Validation can be a significant expense that must trade off with design target value (mean resistance).

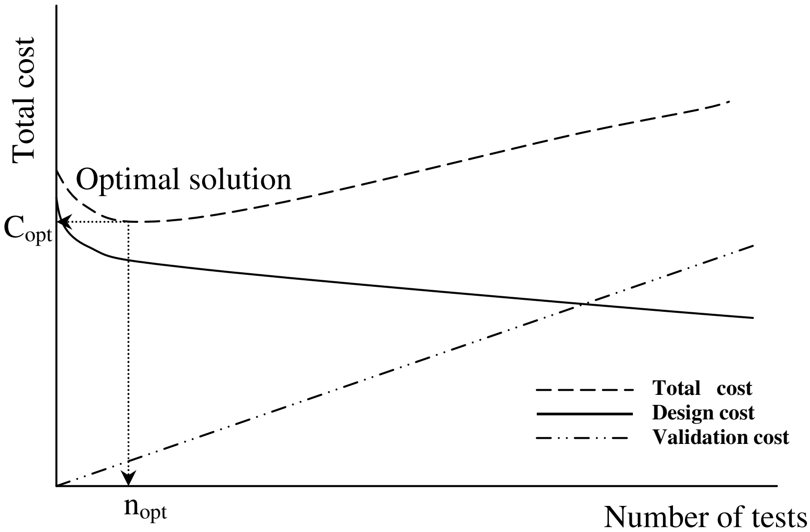


Figure II. 19 design and validation cost

The sum of those two costs in figure II.19 looks as a U- shaped curve with a minimum total cost. However, in this case we can define the optimization problem for the case of limit state function $G(R,S) = R - S$ as following,

$$\min C_{tot} = C_{design} + C_{test} \tag{II.43}$$

under

$$\frac{m_{R_{obj}} - m_s}{\sqrt{(c_R m_{R_{obj}})^2 + \sigma_s^2}} \geq \beta_t \tag{II.44}$$

$$\frac{m_{\bar{R}} - m_{R_{obj}}}{\frac{c_R m_{\bar{R}}}{\sqrt{n_{test}}}} \geq \beta_\alpha \qquad \text{(II.45)}$$

Where $m_{R_{obj}}$, $m_{\bar{R}}$ is the objective mean resistance and mean value of mean resistance random variable, m_s the stress mean value, β_t , β_α target reliability and the inverse function normal distribution at confidence level (1-α) , σ_S the standard deviation of stress, c_R coefficient de variation for resistance, n_{test} is number of tests.

To solve this problem, optimisation conditions must be satisfied, first reliability target β_t , and confidence level β_α .

Numerical example

Here we introduced a simple example to illustrate the interest of this approach. The stress-resistance interference model is considered $G(R,S) = R - S$ where R is normally distributed with coefficient of variation equal to 0.1 and S has a deterministic value equal to 0.4. The reliability target index $\beta_T = 3.8$ and confidence level 0.95 $\beta_\alpha = \beta_{0.05} = \Phi^{-1}(0.95) = 1.645$. Suppose that the cost of test is $\gamma_{test} = 100$ currency units, the cost of product is $\gamma_p = 10$ currency units, and number of products to be produced is 5000 units.

What is the optimal number of tests to ensure reliability target using compound uncertainties approach?

This problem has been solved using equations (II.43), (II.44) and (II.45) using MathCAD software and the results are presented in figure II.20.

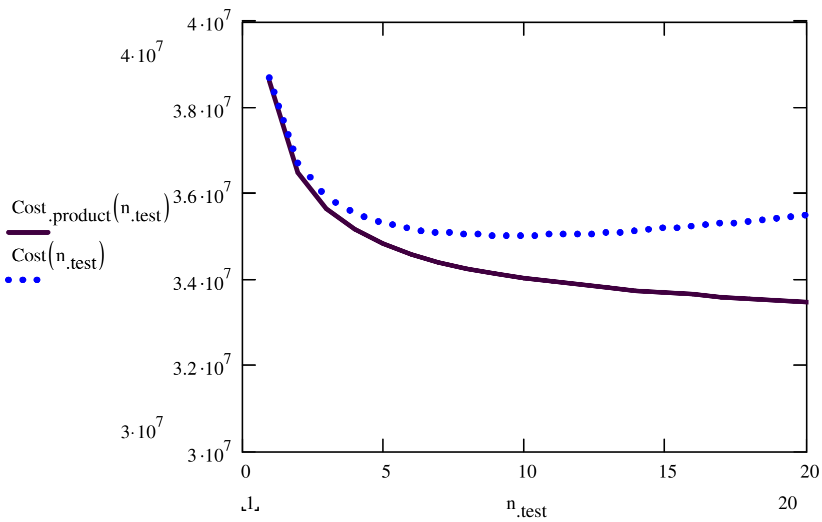


Figure II. 20 Cost versus test number

The optimal number of tests to achieve reliability and validate it is 10 tests with total cost equal to 3.5 x 10^7 currency units.

According to the ratio between the cost of tests and the cost of products the optimal number of tests is obtained for up to 5000 units, in table II.8.

Cost of test/cost of product	Optimal number of tests
10	10
15	7
20	6

Table II. 8 Optimal test numbers versus cost ratio test/product

This means that when the test cost is twenty times the product cost, performing 7 tests is equivalent to the case of performing 10 tests when the test cost equal to ten times the product cost. It is worthy to know what is the minimum number of products is going to be produced to carry on one test, the result presented in table II.9.

Cost of test/cost of product	Minimum number of products
10	135
15	200
20	265

Table II. 9 Minimum production to perform one test according cost ratio (test/product)

It can be shown that the level of reliability target leads to additional costs, due to larger number of tests and larger required resistance. Table II.10 gives the optimal number of tests and the minimum total cost as a function of the target failure probability. When this target goes from 10^{-1} to 10^{-5}, the total cost increases by 50% (i.e. from 2.51 x 10^7 to 3.78 x 10^7) and the testing cost increases by 25% (i.e. from 8 to 10 tests).

Failure probability	Optimal number of tests	Total cost(currency unit)
10^{-1}	8	2.51 x 10^7
10^{-3}	9	3.15 x10^7
10^{-5}	10	3.78 x10^7

Table II. 10 Optimal total costs and test number versus reliability level

The previous analysis enables the designer and supplier to make a judgement for his final choice. This analysis is necessary to decide the chorological plan and testing chamber capacities to avoid lateness and management problems which are losses.

Test number optimisation under reliability and confidence conditions using the compound uncertainties is investigated. This investigation is carried out by applying it to a simple design problem. The results can be obtained from this investigation give the supplier and designer some aspects of guidance which may helps in making a decision regarding the planning and equipment necessary to commence the production process. The above ideas constitutes some adobes must be developed in a general context to be more matured in terms of more complex cases and conditions.

II.8 Conclusion

Reliability demonstration tests is an important tool in the product life cycle, this chapter has introduce this type of testing, illustrating its role and position among the other types of tests. Basically, we have distinguished between two types: predetermined sample size and non-predetermined ones. Bogey tests have a major draw back of large sample size necessary to demonstrate high level of reliability, which is infeasible for certain types of products. The non-predetermined sample size category encompasses sequential and stress resistance ones. Our investigation has been concentrated on four approaches based on structural reliability theory: confidence interval, test hypothesis, Bayesian and compound uncertainties. It was approved by two examples that compound uncertainty method is the best method under the assumptions taken into account. The basic assumption is that the type of probability density function for resistance is known and its. In fact, the most demanding problematic in the industrial world is how to achieve the best products with better characteristics in terms of reliability and quality with lower cost. Such kind of problems is called optimisation problem. Life cycle cost includes several items, initial and validation costs represents an important percentage of the total life cycle. In our work we have considered these two items in a simple optimisation study using compound uncertainties approach.

Solving the cost optimisation equation under the reliability and quality conditions, gives us the optimal number of tests that minimize the major two costs in product life cycle design and validation. Analysing the results of cost optimisation problem enables the designer and supplier tool to find some alternatives in production policy.

Chapter III. Hazard-based design under repetitive loads

III.1 Introduction

As seen before, reliability is defined as the probability that a system will perform properly for a specified period of time under a given set of operating conditions. It is implied in this definition that a clear-cut criterion for failure, from which we may judge at what point the system is no longer functioning properly. Similarly, the treatment of operating conditions requires an understanding of the environment within which it must operate, including the loading to which the system is subjected. Perhaps, the most important effect to which we must relate reliability is the operating time. Therefore, it is in terms of rate of failure that most reliability phenomena are understood. For this purpose, reliability must be considered as a function of time, which leads to the definition of the failure rate. Examining the time dependence of the failure rate allows us to gain additional insight into the nature of failures – whether they are infant mortality failures, failures that occur randomly in time, or failures brought on by aging.

First, in order to lay out the problem of interest, we describe the stress-resistance problem in figure III.1, according to the type of stress, resistance, causes of failure and mechanisms of failure. We can recognize single or repetitive loading, non-degraded and degraded component resistance (when resistance distribution does not change with time or load application, is said to be non-degraded component), and failure causes due to loading or degradation.

This chapter describes the probabilistic design based on stress-resistance model in terms of hazard target rather than failure probability target for the case of repetitive loading and non-degraded resistance, (grey blocs in figureIII.1). The case of degraded product is considered in the fourth chapter. Our concern here is the repetitive loading, because the failure may take place in the useful period of product life cycle due to loading variability [Lew-94, Car-97].

III.2 Types of loading

Generally, loads include imposed displacement and temperature effects as well as forces and moments; they have often much greater uncertainty than resistance. They may arise from uncertain environmental conditions such as winds, snow, ice and waves. They may also vary with time in magnitude, position or induce dynamic response. In figure III.2, load modelling is classified where we can distinguish two types: deterministic and probabilistic. In reality, there is no meaning of deterministic loads, because of uncertainties related to the load and its environment. We distinguish between two types of probabilistic loads: single and repetitive. Single load has been usually considered in structural reliability theory. Concerning repetitive loads, four basic approaches may be applied:

- Extreme value distributions
- Peaks over threshold (POT Gumbel)
- Homogenous and non-homogenous Poisson processes.
- Hazard based design.

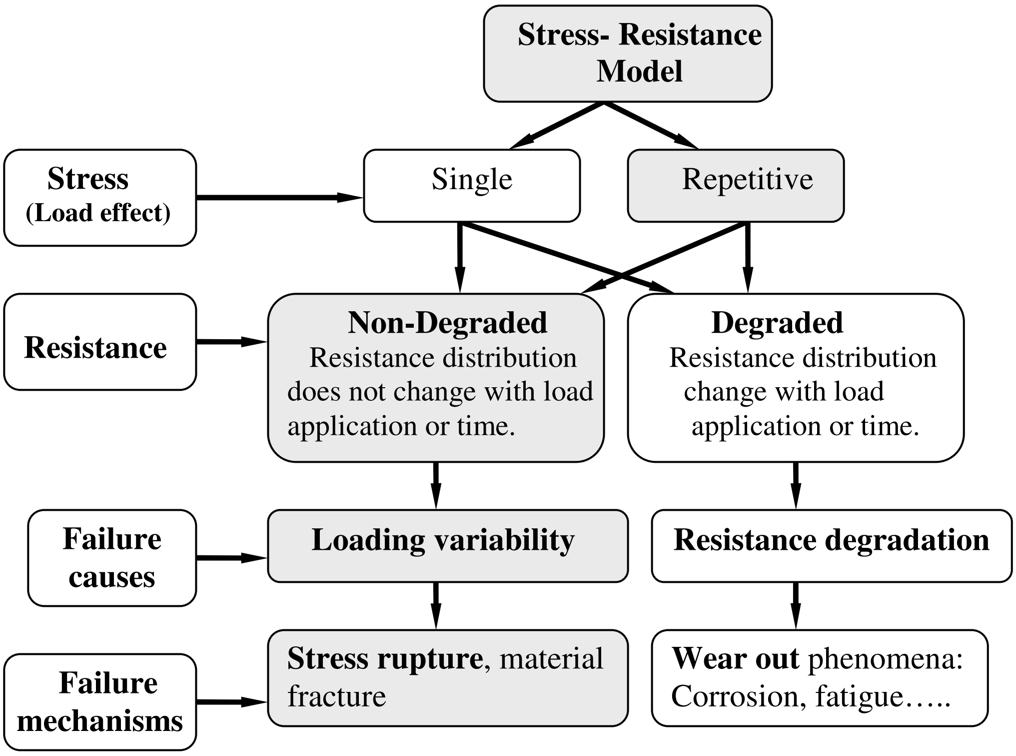


Figure III. 1 Stress-resistance classification of design problem

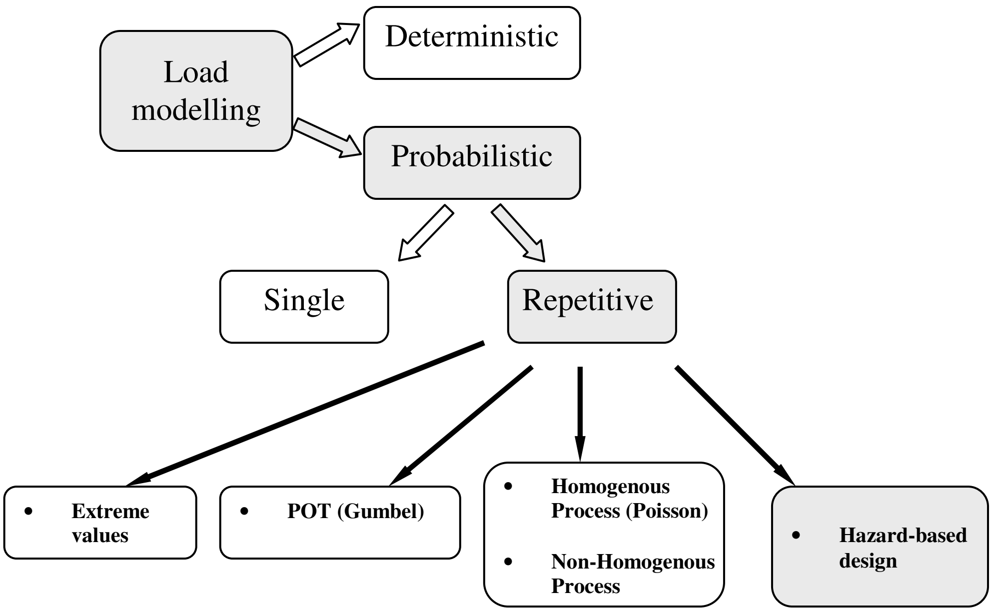


Figure III. 2 Load modelling

III.2.1 Probabilistic modelling of loading

III.2.1.1 Single loading

The applied load on a system corresponds to the maximum load from the beginning of load application until its removal. Figure III.3 adapted from [Lew-94], indicates the time dependence of some loading patterns that may be treated as single loading. Figure III.3 (a) represents a single loading of finite duration, as examples for this kind of loading; we have missiles during lunch or the applied torque on bolts. Figure III.3 (b) shows series shocks which would be typical of vibration loading, earthquakes and impact loading on aircraft during landing. In these two kinds of loading (*a, b*) the duration is short enough that no weakening of the system capacity takes place. If no decrease in system capacity is possible, the situations shown in Figures III.3(c) and *d* may be considered as single loadings, even though they are not of finite duration. The loading shown in figureIII.3(c) is typical of dead loads related to the own weight of the structure; these loads increase during construction and then remain at constant level. Fig III.3 (d) may be viewed as a single loading. Provided the peaks of the same magnitudes, the system will either fail the first time the load is applied or will not fail at all. But under cyclic loading, there will be decrease in resistance. Metal fatigue and other wear out effects are likely to weaken the resistance of the system gradually. Now, if the value of peak magnitudes varies from cycle to cycle, we must consider the time dependence as in the case of Load variability.

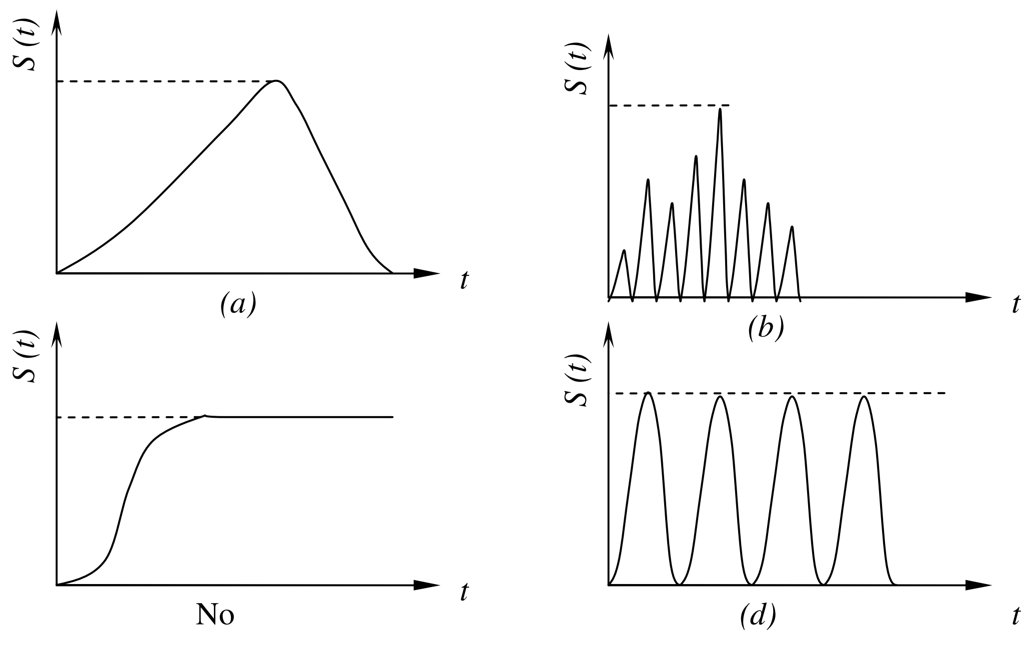


Figure III. 3 Time dependent loading patterns

III.2.1.2 Repetitive loading

Our concern in this chapter is the random failures resulting from random repetitive loads.
In stress-resistance interference theory, with longer exposure time the load distribution would shift to the right, causing the reliability to decrease likewise aging effects. Therefore it is quite important to investigate how repetitive load can be modelled.

III.2.1.2.1 Extreme value distributions

The maximum applied load on a component will often occur in extreme conditions. For marine structures, such loads may arise from waves, winds currents, or some combination of these. At the design stage, the magnitude of the largest load is random. We can predict the 'most likely' maximum load but there will be large uncertainty about whether the real applied load will be somewhat greater or less than this value. The extreme value distribution allows us to characterise this uncertainty.

For the case of the non-degraded systems, resistance is considered constant (i.e. no significant degradation) as shown in figure III.4. The probability of failure is given by:

$$P_f(T) = \Pr[S_{\max}(T) \geq R] \tag{III.1}$$

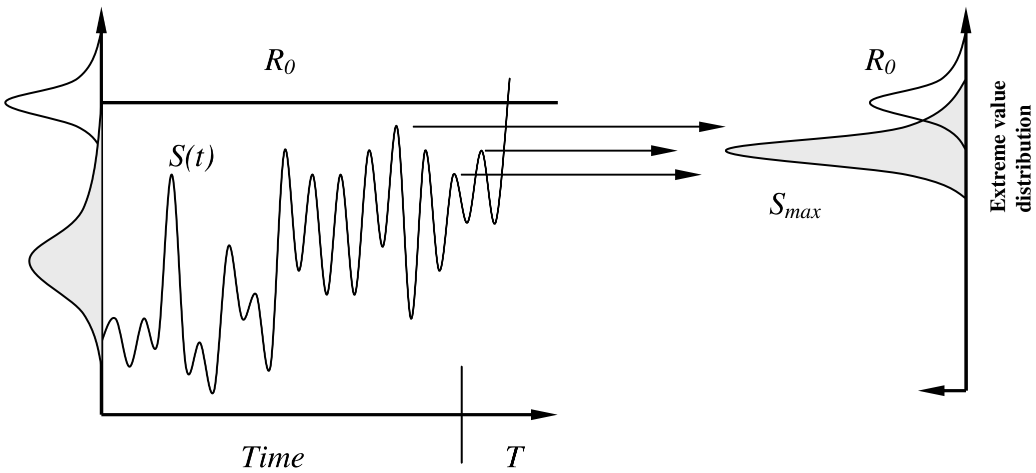


Figure III. 4 Integration of the loading history

For a large number of load applications, the theory of extreme value gives a very good approximation of the product reliability. Under the assumption of independent applications of the load, the probability that the maximum value $S_{\max}$ is lower than a certain value x is:

$$F_{S_{\max}(x)} = \Pr[S_{\max} < x] = \Pr[S_1 < x] \cdot \Pr[S_2 < x] \cdots = (F_S(x))^n \tag{III.2}$$

where S_i is the load at the i^{th} application. For n very large, this distribution tends toward the extreme value distribution. Hence, the probability of failure under a number of loads lower or equal to n is calculated by:

$$P_f = \Pr[N \leq n] = \int_0^{\infty} 1 - [F_S(x)]^n \; f_R(x)\; dx == \int_0^{\infty} (1 - F_{S\max}(x)) \; f_R(x)\; dx$$

Or:

$$P_f = \int_0^{\infty} f_{S\max}(x) \; F_R(x) \; dx$$

This approach allows us to convert the time dependent problem into a time independent one, under the cost of introducing certain conservatism (more or less significant). Nevertheless, this technique is not adapted for all load combinations, because it is based on the assumption of the independent and simultaneous occurrence of all the maximum values of the various loads, which is far from being realistic. The theory of peaks over threshold (POT) gives the solution for this deficiency.

III.2.1.2.2 Peaks over threshold (POT) approach

This approach has been used in flood frequency studies to estimate the required height of river and coastal defences. The basic idea is shown in figure III.5. If these values are plotted as a histogram, we just try to model the values in the upper tail above the threshold y_0 (figure III.5) with suitable distribution.

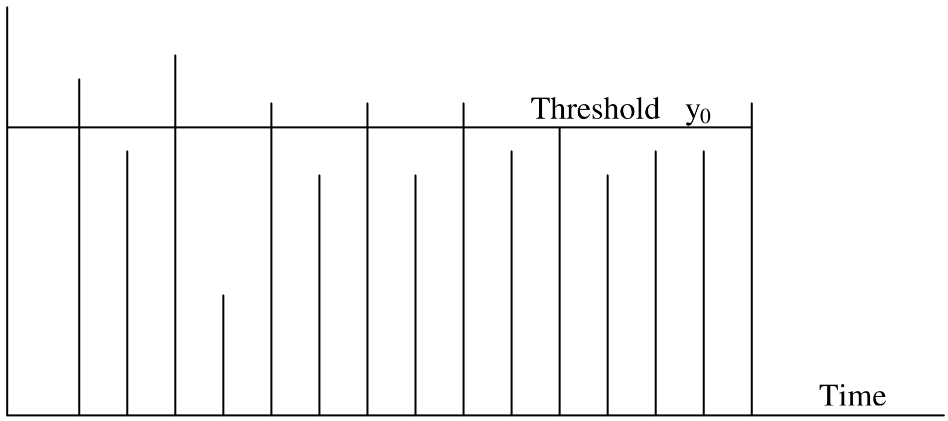


Figure III. 5 Series of flood peak heights, out-crossing the threshold.

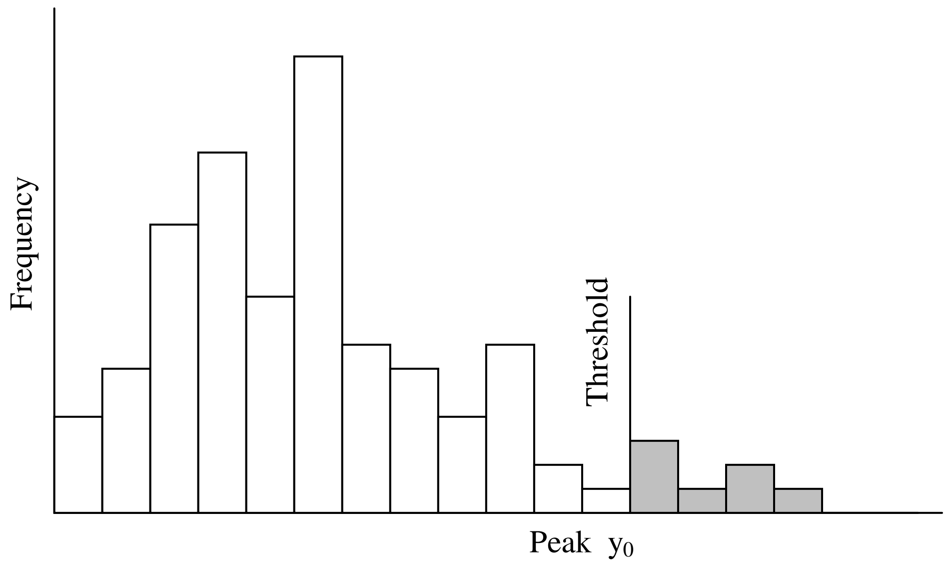


Figure III. 6 Histogram of all flood peaks

Although many distributions could be suitable, the exponential one is the most widely used. For the tail region, the exponential CDF is given by:

$$F_y(y) = 1 - \exp(-\lambda(y - y_0)) \qquad \text{valid} \quad \text{for } y > y_0 \qquad \text{(III.3)}$$

Let us now consider the values y_1, y_2, y_3 .... all greater than y_0. Then by definition $\Pr[y_i < y] = F_y(y)$ and hence:

$$\Pr[(y_1 < y)\cap(y_2 < y)\cap(y_3 < y)\cap.......(y_n < y)] = [F_y(y)]^n \qquad \text{(III.4)}$$

If the largest of all the y_i , satisfies the conditions: $y_{max} <y$, then all the other values must be less than y; hence:

$$\Pr[y_{\max} < y] = [F_y(y)]^n = F_{y\max}(y)$$

In this case:

$$F_{y_{\max}}(y) = [1-\exp(-\lambda(y-y_0))]^n = \left[1 - \frac{n\exp(-\lambda(y-y_0))}{n}\right]^n \qquad \text{(III.5)}$$

Let $t = n\exp(-\lambda(y-y_0))$, then $F_{y_{\max}}(y) = \left[1-\frac{t}{n}\right]^n \approx \exp(-t)$ for a large n as the series expansion for both sides are identical for the first two terms, i.e. $\left[1-\frac{t}{n}\right]^n = 1-t+....$ and e^{-t} =*1*- *t* +......It becomes: $F_{y_{\max}}(y) = \exp[-n\exp(-\lambda(y-y_0))]$.

As n=exp (ln(n)), this expression can be written:

$$F_{y_{\max}}(y) = \exp[-\exp(-\lambda(y-y_0)+\ln n)] \qquad \text{(III.6)}$$

Defining $\lambda = \alpha_n$ and $u_n = y_0 + \ln n/\lambda$ we obtain the expression

$$F_{y_{\max}}(y) = \exp[-\exp(-\alpha_n(y-u_n))]$$

By differentiation, we get the expression for $f_{max}(y)$:

$$F_{y_{\max}}(y) = \alpha_n\, e^{-\alpha_n(y-u_n)} \exp[-\exp(-\alpha_n(y-u_n))] \qquad \text{(III.7)}$$

This is called the maximum extreme value Type I (EV1), Gumbel TypeI or Fisher-Tippet Type I. The extreme value distribution can be applied to any underlying distribution that has an exponential tail. Gumbel classified these distributions into three categories: Types I (exponential doubles), type II (exponential) and type III (exponential with upper limit). For a large number *n*, Gumbel showed that the distribution of the extreme values does not depend on the exact form of the underlying distribution, but it depends primarily on the shape of the tail of this distribution. In other words, the central part of the underlying distribution has a little effect on the form of the extreme values distribution.

III.2.1.2.3 Homogenous Poisson process loading variability

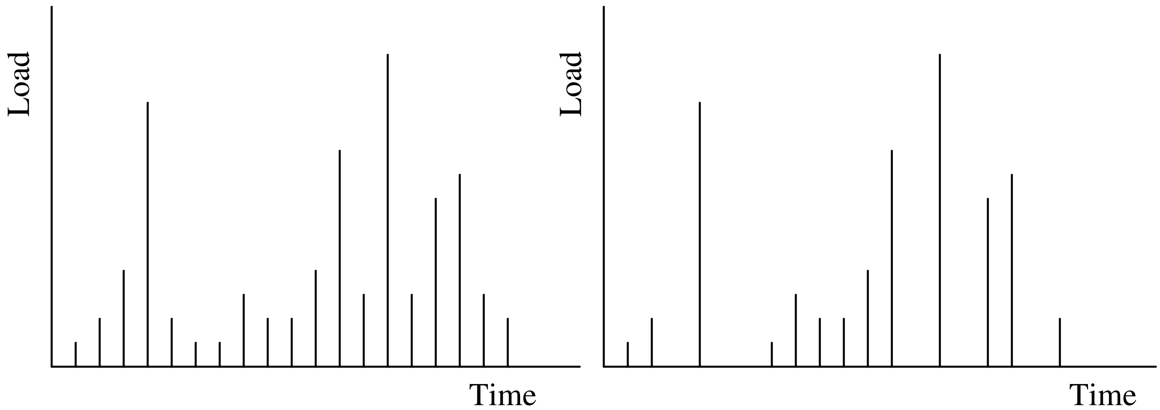
In this model, certain assumptions must be considered:

1. There is ordinarily no load on the component. The load applications occur instantaneously at random time intervals governed by a homogenous Poisson process with intensity γ;
2. The load duration is negligible.
3. The load magnitudes are independent with CDF $\boldsymbol{F_s(.)}$ **.**
4. The resistance is random variable with PDF, $f_R(.)$.

Suppose that we specify a component with a known resistance *R(t)* at time *t,* the probability that a load occurring at time *t* will cause system failure is just the probability that $S > R(t)$, or

$$p = \int_{R(t)}^{\infty} f_S(s)ds$$

The repetitive loading may occur at either equal or random time intervals (figure III.7.a-b); in our case, it is assumed that it is random. The loading is Poisson distributed in time with frequency γ (i.e. *the probability of load occurring is independent of the time at which the last loading occurred*).



(a) Periodic loading (b) Loading at random intervals

Figure III. 7 repetitive loads of random magnitudes.

Assuming that loads occur during a vanishing small time increment Δt, the probability of load occurrence is $\gamma.\Delta t$, with ($\gamma.\Delta t << 1$). The probability to have a load large enough to cause failure within[t, $t+\Delta t$] is thus :

$$p\ \gamma\ \Delta t = \gamma \int_{R(t)}^{\infty} f_S(s)\, ds\, \Delta t$$

The system will fail between *t* and *t*+ Δt if it survived to time *t* and failure will occur during Δt. As $\Re(t)$ is the probability to survive till *t,* the failure probability during Δt is $\Re(t)p\gamma\Delta t$. Similarly, the reliability at $t+\Delta t$ is that the probability that the system has survived to t and that no failure occurs during Δt.

$$\Re(t+\Delta t)=\left[1-\gamma\int_{R(t)}^{\infty} f_S(s)ds.\Delta t\right]\Re(t)$$

Therefore:

$$\frac{\Re(t+\Delta t)}{\Delta t}=\gamma\int_{R(t)}^{\infty} f_S(s)ds\,\Re(t)$$

$$h(t)=-\frac{1}{\Re(t)}\frac{d}{dt}\Re(t)$$

$$h(t)=\gamma\int_{R(t)}^{\infty} f_S(s)ds$$

III.3 Repetitive load design based on hazard

The limit state function is given by: $G(R,S)=R-S$ (figure III.8). For the case of independent normal variables, we recall the failure surface in equation I.18 after making the probabilistic transformation.

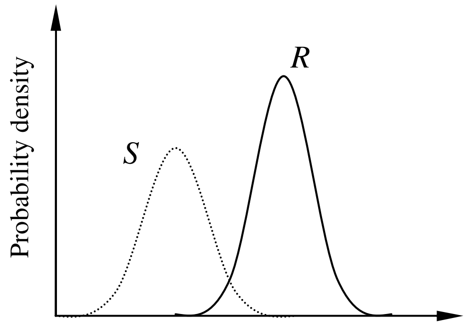


Figure III. 8 load strength interference model.

The expression in the normalized space is:

$$H(U_1,U_2)=m_R-m_S+\sigma_R U_1-\sigma_S U_2 \tag{III.8}$$

The minimum distance from the origin to the failure surface is given by:

$$\frac{m_R-m_S}{\sqrt{\sigma_R^2+\sigma_S^2}} \tag{I.17}$$

As the failure condition is not modified by any proportional coefficient, we can divide the above the above equation by $\sqrt{\sigma_R^2+\sigma_S^2}$:

$$H(U_1^*,U_2^*)=\frac{m_{R_1}-m_S}{\sqrt{\sigma_R^2+\sigma_S^2}}+\frac{\sigma_R}{\sqrt{\sigma_R^2+\sigma_S^2}}U_1^*-\frac{\sigma_S}{\sqrt{\sigma_R^2+\sigma_S^2}}U_2^*$$

At the most probable point, this equation takes the form

$$H(u_1^*, u_2^*) = \beta + \alpha_R u_1^* - \alpha_S u_2^* = 0 \quad \text{(I.18)}$$

where α_R and α_S are the directional cosines of resistance and load respectively. From equation I.18, the design problem depends on the reliability index β, and the direction cosines of resistance and load α_R, α_S. However, for the case of non-degraded components, the load sensitivity α_S has often the key role in reliability, as the load variability is usually much larger than resistance variability. As mentioned in chapter I, the load sensitivity α_S is called the loading roughness by Carter [Car-97].

The probabilistic design methodologies, based on stress-resistance interference model, consider the failure probability as a design target to be achieved for single load application. As a matter of fact, the target "probability of failure" varies with the number of load applications. For a large number of loads the case of repetitive loads is treated by extreme value. However, for moderate number of load applications the design won't be robust (un-sensitive to the number of load applications) unless the previous knowledge of the exact number of load applications along the product life cycle.

In the present work, hazard is proposed as a design target instead of failure probability, because of its robustness with regard to our concerned problem (repetitive loads and non-degraded components) figure (III.8).

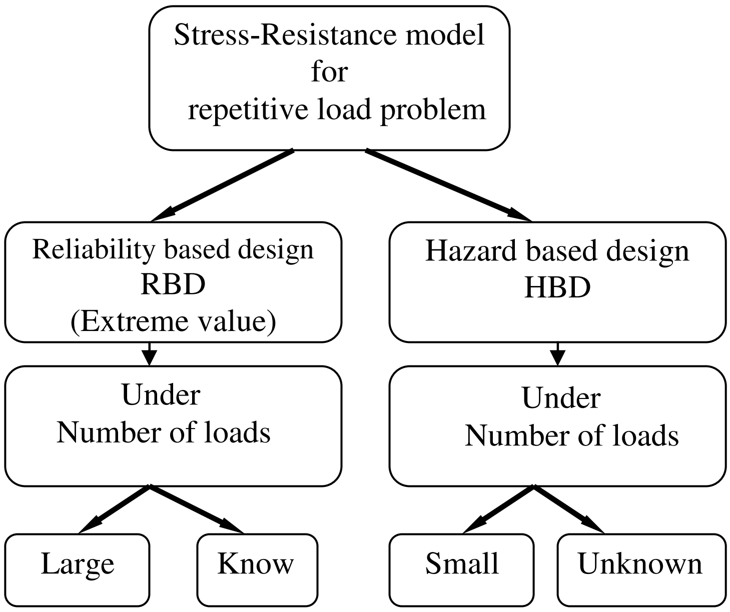


Figure III. 9 Repetitive loading and stress-resistance model.

To verify the robustness of the proposed criterion, sensitivity analysis is carried out for both hazard and reliability in the following subsections.

III.3.1 Sensitivity analysis

Reliability is measured by the probability of survival of the component under *n* load applications. The general expression for reliability, from stress–resistance model under repetitive loads is given by:

$$\Re(n) = \int_{-\infty}^{\infty} f_R(s)\left[\int_0^s f_S(s)ds\right]^n ds = \int_{-\infty}^{\infty} f_R(s)[F_s(s)]^n ds \quad \text{(III.9)}$$

For independent normal distributions of resistance and stress equation (III.9) becomes

$$\Re(n) = \int_{-\infty}^{\infty} f_R(s)\left[\Phi\left(\frac{s-m_S}{\sigma_S}\right)\right]^n ds = \int_{-\infty}^{\infty} \frac{1}{\sigma_R\sqrt{2\pi}}\exp\left[-\frac{(s-m_R)^2}{2\sigma_R^2}\right]\left[\Phi(\frac{s-m_S}{\sigma_S})\right]^n ds \quad \text{(III.10)}$$

$$\Re(n) = \frac{1}{\sigma_R\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{s-m_R}{\sigma_R}\right)^2\right]\left[\Phi\left(\frac{s-m_S}{\sigma_S}\right)\right]^n ds \quad \text{(III.11)}$$

Substitute $u = \dfrac{s-m_R}{\sigma_R}$ leads to:

$$\Re(n) = \int_{-\infty}^{\infty}\phi(u)\left[\Phi\left(\frac{u-\dfrac{(m_S-m_R)}{\sigma_R}}{\dfrac{\sigma_S}{\sigma_R}}\right)\right]^n du \quad \text{(III.12)}$$

where $\phi(u)$ is standard normal PDF the probability density function of and $\Phi()$ is the normal CDF with mean value equal to $\dfrac{(m_S-m_R)}{\sigma_R}$ and standard deviation equal to $\dfrac{\sigma_S}{\sigma_R}$. The integral (III.12) can be expressed in terms of beta index β, and load sensitivity α_S.

$$\Re(n) = \int_{-\infty}^{\infty}\phi(u)\left[\Phi\left(\frac{u-\dfrac{-\beta}{\sqrt{1-\alpha_s^2}}}{\dfrac{\alpha_s}{\sqrt{1-\alpha_s^2}}}\right)\right]^n du \quad \text{(III.13)}$$

Failure probability is therefore:

$$F(n) = 1-\int_{-\infty}^{\infty}\phi(u)\left[\Phi\left(\frac{u-\dfrac{-\beta}{\sqrt{1-\alpha_s^2}}}{\dfrac{\alpha_s}{\sqrt{1-\alpha_s^2}}}\right)\right]^n du \quad \text{(III.14)}$$

For our case, the normal CDF has a mean value equal to $\dfrac{-\beta}{\sqrt{1-\alpha_S^2}}$, and its standard deviation equal to $\dfrac{\alpha_s}{\sqrt{1-\alpha_s^2}}$. In this case, the reliability is a function to the following variables *n*, β and α_S.

The hazard equation presented in *chapter I* in terms of reliability or failure probability.

$$h(n) = \frac{F(n) - F(n-1)}{1 - F(n)} \quad or \quad h(n) = \frac{\Re(n-1) - \Re(n)}{\Re(n)} \quad n \geq 2 \tag{III.15}$$

By differentiating the hazard function with respect to the number of loads we find:

$$\frac{\partial h}{\partial n} = \frac{(F'(n) - F'(n-1))(1 - F(n)) - (F'(n))(F(n) - F(n-1))}{(1 - F(n))^2}$$

which cab be arranged as:

$$\frac{\partial h}{\partial n} = \frac{F'(n)[1 - F(n-1)] - F'(n-1)[1 - F(n)]}{(1 - F(n))^2}$$

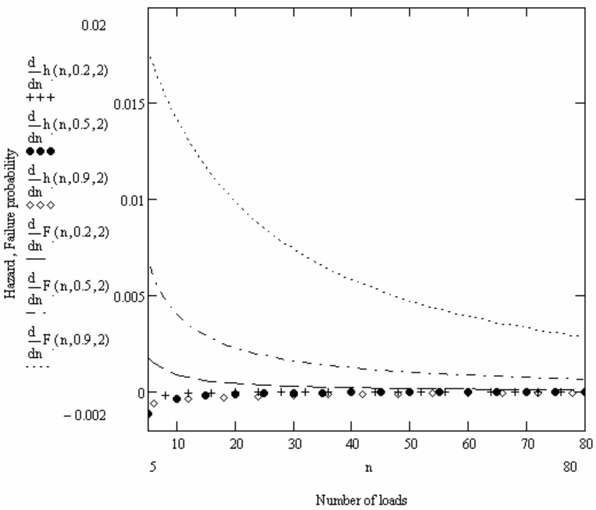
Knowing that $F(n) <<< 1$ for engineering components or systems, the above expression can be approximated as: $\frac{\partial h}{\partial n} \approx F'(n) - F'(n-1) \ < \ F'(n)$

This leads to

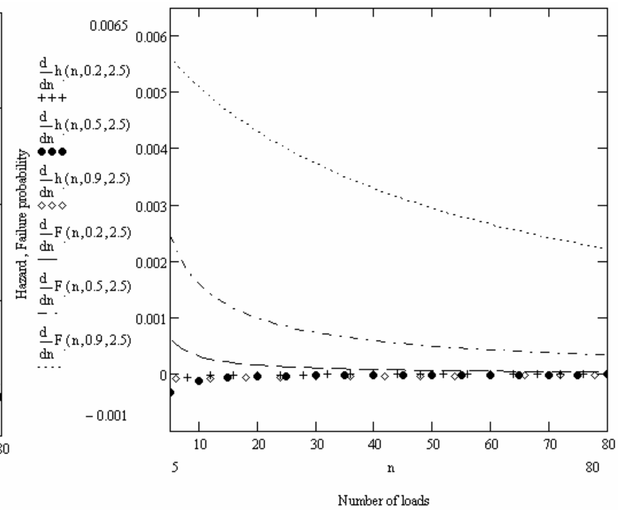
$$h'(n) < F'(n) \quad \text{as long as} \quad F'(n) > 0 \ \text{ and } \ F'(n-1) > 0$$

We can conclude that hazard behaviour in the case of repetitive loads and non-degraded components, is less sensitive than failure probability especially, in the case of rough loads.

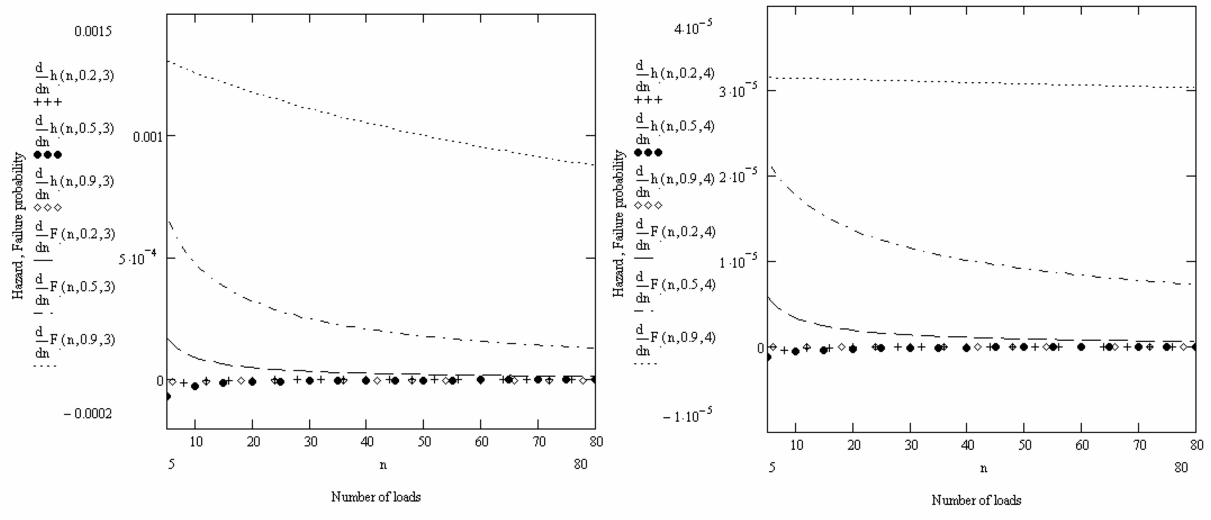
To make a comparison between the sensitivity of both failure probability and hazard with regard to the number of loads, we have plotted in figure III.10.a-d, the derivatives of hazard h and failure probability F, for smooth, medium and rough loads (i.e. $\alpha_s = 0.2$, 0.5 and 0.9) at different levels of safety index (i.e. $\beta = 2,\ 2.5,\ 3$ and 4).



$(a)\quad \beta = 2$



$(b)\quad \beta = 2.5$



$(c)\quad \beta = 3$ $(d)\quad \beta = 4$

Figure III. 10 hazard and failure probability sensitivity

Although the failure probability varies strongly with the number of loads, hazard shows a very low sensitivity to the number of loads, despite of changing reliability index β and loading roughness. With respect to the reliability index β, the failure probability shows poor convergence, high sensitivity, at low level of safety margin (i.e. $\beta = 2$) this convergence becomes worse with the increase of direction cosine of load. At a constant level of direction cosine, the convergence improves with the increase of safety margin.

At $\beta = 2$ Table III.1 gives the derivatives of hazard $\partial h/\partial n$ and failure probability $\partial F/\partial n$ and for $\alpha_s = 0.2, 0.5,\ 0.9$ at the number of loads values *n=10, 20 and 50*. Hazard sensitivities remain stable in various cases, its values fall in the 10^{-4} range for the rough loads and reaches 10^{-5} values for smoother ones. However, the sensitivity of failure probability decreases with the increase of the number of load applications for the case of rough loads (i.e. $\alpha_s = 0.9$) (Table III.1 a). This sensitivity has lower values as the load becomes smoother (table III.1 b).

$\beta = 2$ / $\partial F, \partial h/\partial n$	(a) $\alpha_s = 0.9$		
	n =10	*n =20*	*n =50*
$\partial F/\partial n$	0.014	0.01	4.73 10-3
$\partial h/\partial n$	-4.36 10^{-4}	-2.56 10^{-4}	-9.88$10^{-5}$
$\beta = 2$ / $\partial F, \partial h/\partial n$	(b) $\alpha_s = 0.5$		
	n =10	*n =20*	*n =50*
$\partial F/\partial n$	4.04 10^{-3}	2.30 10^{-3}	1.03 10-3
$\partial h/\partial n$	-3.04 10^{-4}	-9.92 10^{-5}	-2.04 10^{-5}
$\beta = 2$ / $\partial F, \partial h/\partial n$	(c) $\alpha_s = 0.2$		
	n =10	*n =20*	*n =50*
$\partial F/\partial n$	8.99 10^{-4}	4.48 10^{-4}	1.79 10^{-4}
$\partial h/\partial n$	-1.04 10^{-4}	-2.47 10^{-5}	-3.85 10^{-5}

Table III. 1 Numerical sensitivity

For this reason, we believe that setting hazard as a design target is more robust than failure probability under the following two conditions:

- Non-degraded components.
- Small or unknown repetitive loads.

III.3.2 Hazard-based design for independent normal variables

Let us consider the case of two independent normally distributed variables *R* and *S*. For a given acceptable level of hazard as a design target, we have to carry out the design with the following available data:

- Stress *S*, defined by a normal distribution with known parameters (i.e. mean value m_s, and standard deviation σ_s).
- Resistance *R,* defined by normal distribution, where only the coefficient of variation is known c_R.

The aim of the design is to find the resistance mean, which satisfies the hazard target h_t. In other words, we search for β_H the reliability index corresponding to hazard target h_t.We have seen that hazard is a function of the following variables: β , α_s and *n*. The typical hazard curve obtained by plotting hazard versus reliability index at specified load sensitivity is presented in (figure III. 11 a).

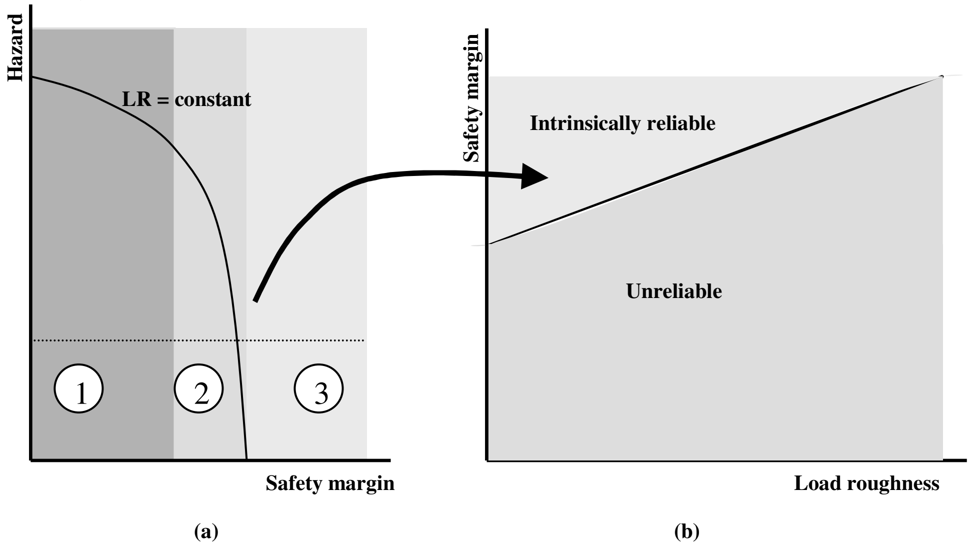


Figure III. 11 a) Characteristic regions of typical hazard-safety margin, b) safety margin-Loading roughness at negligible hazard.

The curve consists of three regions

1) ***Region 1***: the hazard is too high for practical use (low safety margin).
2) ***Region 2***: in this region, we have acceptable values of hazard but hazard is too sensitive to design parameters, according to Taguchi philosophy in his work on experiment design: *"Designs whose reliabilities (or any other quality criterion) are sensitive, to any uncontrollable design parameter is un-acceptable from a quality point of view*". That means; designs must be insensitive to uncontrollable parameters. If the designs satisfy this we call them "robust".
3) ***Region 3***: in this region negligible hazard (Zero hazards), this region is said to be intrinsically reliable.

To solve sensitivity problem, Carter [Car-97] suggested truncating the curve at a negligible limiting value of hazard equal to 10^{-9} (figure III.11. a).The limit values of safety margin at different loading roughness are plotted in (figure III.11. b); this curve is called "the design curve". The designer can use the design curve to obtain intrinsically reliable system, just by reading off β at specified value of α_S.

To conclude there are two issues in this procedure:

I) Design for a negligible hazard as a target (intrinsically reliable).

II) There is no universal design curve. Design curve must be established for any combination of *S,* and *R* and at any target hazard level we decide.

The design problem becomes an optimisation problem

$$\left\| \begin{array}{l} Find \quad \beta_H \\ \\ under \quad h(n,\beta,\alpha_S) - h_t \leq 0 \end{array} \right.$$

where β_H is the safety margin corresponded to hazard target, $h(n,\beta,\alpha_s)$ is the hazard equation (III.15) obtained according to the integral (III.13).

Solution procedure

In this subsection, we give the solution procedure for hazard-based design. Given that both probability densities of load and resistance are known and the acceptable level of hazard. Then, required data for proceeding design are:

- Hazard target for design
- Statistical parameters for both load and material resistance
- The convergence tolerance in calculating the reliability index. If the required data mentioned before are available then our goal is to find the resisting strength m_R to provide intrinsically reliable design.

The procedure to perform design calculation is as following figure III.12:

1) Initial guess value for safety margin β_i.

2) Calculate the $k(\beta_i)$ resultant and therefore the mean value of resisting strength m_{R_i}.

3) Calculate the standard deviation value of m_{R_i} using coefficient of variation of resistance.

4) Calculate α_{s_i} value.

5) Use α_{S_i} to obtain a new value of reliability index β, $\beta_H(\alpha_{s_i})$.

6) Check for convergence. If accurate go to step 7 otherwise find another initial guess value. The average of the two values is the new guess value, and go to step 2.

7) If convergence is satisfied, the design parameters are $\alpha_s = \alpha_{s_i}$, $\beta = \beta_i$, $m_R = m_{R_i}$,

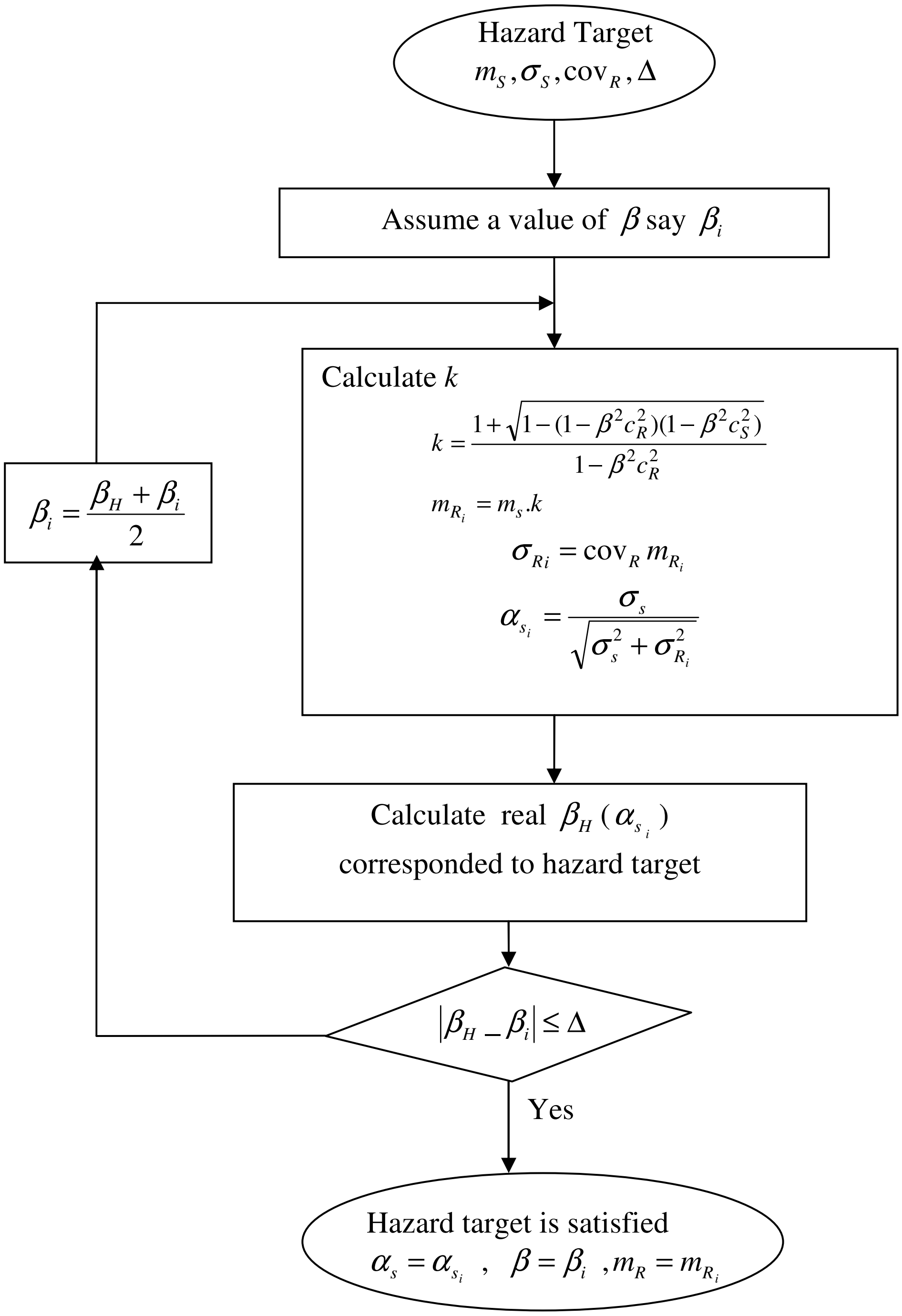


Figure III. 12 Flow chart of hazard based design

III.3.3 lognormal distributions case

When R and S are log-normal distributions, the logarithms $\ln R$ and $\ln S$ follow the normal distribution, with parameters:

$$\lambda_R = \ln\frac{m_R}{\sqrt{1+c_R^2}}, \zeta_R = \sqrt{\ln(1+c_R^2)}$$

$$\lambda_S = \ln\frac{m_S}{\sqrt{1+c_S^2}}, \zeta_S = \sqrt{\ln(1+c_S^2)}$$

In this case, we can write the limit state function under the form: $G = \ln R - \ln S$

Given that the function ln is increasing in monotony. The mean value and standard deviation of this margin are:

$$m_G = \lambda_R - \lambda_S$$

$$\sigma_G = \sqrt{\xi_R^2 + \xi_S^2}$$

Since G follows normal distribution the safety margin or reliability index can be written:

$$\beta = \frac{\lambda_R - \lambda_S}{\sqrt{\xi_R^2 + \xi_S^2}} = \frac{\ln\frac{m_S}{\sqrt{1+c_S^2}} - \ln\frac{k\, m_S}{\sqrt{1+c_R^2}}}{\sqrt{\ln(1+c_R^2) = \ln(1+c_S^2)}}$$

It becomes:

$$\ln k + \ln\frac{m_s}{\sqrt{1+c_R^2}} = \ln\frac{m_s}{\sqrt{1+c_S^2}} + \beta\sqrt{\ln(1+c_R^2) + \ln(1+c_S^2)}$$

$$\ln k = \ln\frac{\sqrt{1+c_R^2}}{\sqrt{1+c_S^2}} + \beta\sqrt{\ln(1+c_R^2) + \ln(1+c_S^2)}$$

The safety factor is simply calculated from β and coefficient of variation by:

$$k = \exp\left[\ln\frac{\sqrt{1+c_R^2}}{\sqrt{1+c_S^2}} + \beta\sqrt{\ln((1+c_R^2)(1+c_S^2))}\right] \quad \text{(III.16)}$$

In this case, we can use the same approach described in figure (III.13), with new value of k equation (III.16), and by substituting λ_S, λ_R and ξ_S, ξ_R instead of m_R, m_S and σ_S, σ_R respectively.

III.3.4 Weibull distributions Case

Due to the limitations of normal distributions in modelling resistance and stress, the approach can be extended to other distributions; we have chosen Weibull distribution to model both of stress and resistance. Weibull distribution has the following properties:

- It can be used to represent a wide range of distributions as illustrated in figure III.13.
- It can be used for vast majority of failure patterns.
- It can represent limited lower tail distributions unlike the unlimited distributions (such as the normal distribution).

With the shape parameter β_W =3.44 the median equal, to the mean in Weibull distribution and approximate the normal distribution.

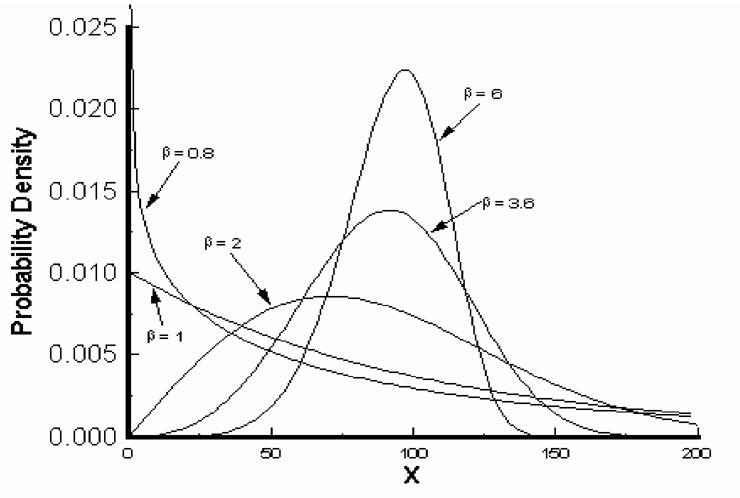


Figure III. 13 Weibull distribution for different shape parameters

From reliability under repetitive loads (equation III.9), the application of Weibull PDF for load and resistance leads to

$$\Re(n)=\frac{\beta_{WR}}{\eta_R}\int_{\tau_S}^{\infty}\left(\frac{x-\tau_R}{\eta_R}\right)^{\beta_{WR}^{-1}}\exp\left[-\left(\frac{x-\tau_R}{\eta_R}\right)^{\beta_{WR}}\right]\left[1-\exp\left[-\left(\frac{x-\tau_S}{\eta_S}\right)^{\beta_{WS}}\right]\right]^n dx \qquad \text{(III.17)}$$

where, β_{WS} , β_{WR} , η_S , η_R , τ_S , τ_R are shape, scale , and initial offset parameters for stress and resistance respectively.

$$\Re(n)=\frac{\beta_{WR}}{\eta_R}\int_{\tau_S}^{\infty}\left(\frac{x-\tau_R}{\eta_R}\right)^{\beta_{WR}^{-1}}\exp\left[-\left(\frac{x-\tau_R}{\eta_R}\right)^{\beta_{WR}}\right]\left[1-\exp\left[-\left(\frac{x-\tau_S}{\eta_S}\right)^{\beta_{WS}}\right]\right]^n dx \quad \text{(III.18)}$$

Substitute $u_R=\frac{x-\tau_R}{\eta_R}$, $A=\frac{\eta_R}{\eta_S}$, $B=\frac{\tau_R-\tau_S}{\eta_S}$

$$\Re(n)=\beta_{WR}\int_{0}^{\infty}u_R^{\beta_{WR}^{-1}}.\exp(-u_R^{\beta_{WR}}).\left[1-\exp\left[-\left(Au_R+B\right)^{\beta_{WS}}\right]\right]^n du_R \qquad \text{(III.19)}$$

The mean and standard deviation for Weibull distribution can be found using the method of moment's theory as a function of Gamma function. For resistance:

$$m_R=c_R.\eta_R+t_R \quad , \quad \sigma_R=d_R.\eta_R \qquad \text{(III.20)}$$

and the mean and standard deviation for Load

$$m_S=c_S.\eta_S+t_S \quad , \quad \sigma_S=d_S.\eta_S \qquad \text{(III.21)}$$

Where,

$$c_S=\Gamma\left(1+\frac{1}{\beta_{WS}}\right) \quad , \quad d_s=\sqrt{\Gamma\left(1+\frac{2}{\beta_{Ws}}\right)-\Gamma\left(1+\frac{1}{\beta_{Ws}}\right)^2}$$

$$c_R=\Gamma\left(1+\frac{1}{\beta_{WR}}\right) \quad , \quad d_R=\sqrt{\Gamma\left(1+\frac{2}{\beta_{WR}}\right)-\Gamma\left(1+\frac{1}{\beta_{WR}}\right)^2}$$

where Γ denote the Gamma function. Then, integral variables A and B are expressed in terms of reliability index β and loading sensitivity α_s as following:

$$A = \frac{d_S}{d_R}\sqrt{\frac{1}{\alpha_s^{\,2}} - 1} \quad , \quad B = \beta_H \sqrt{\mathrm{d}_S^2 + (d_R\,\mathrm{A})^2 - (c_R\,\mathrm{A} - c_S)}$$

The rest of the problem is straight forward, following the same steps in the normal distribution case. After defining $h(.)$ function, the Bisection method is used to find the β_H reliability index at the target hazard level.

As an example, the hazard-safety margin and safety margin-load roughness results are presented in figure III.14 at different loading roughness for both normally (upper line) and Weibull distribution (lower line) with shape parameter 3.44 for both resistance and load.

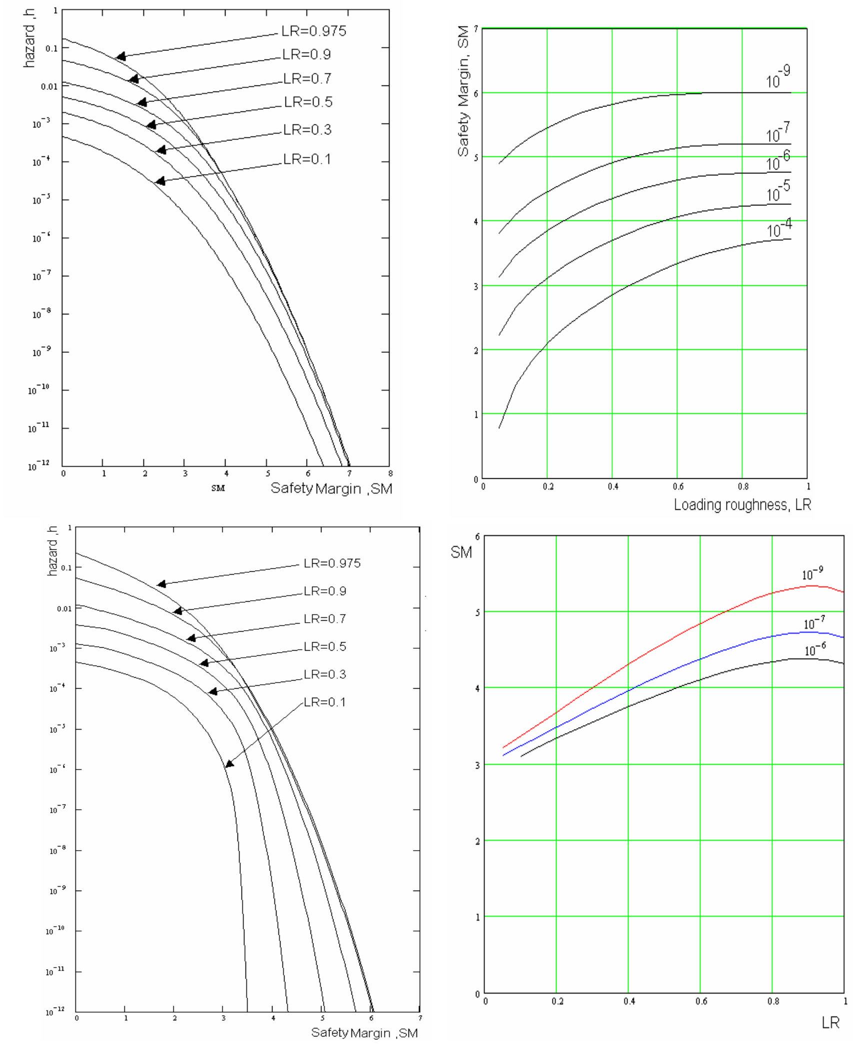


Figure III. 14 H-SM at different values of LR , SM-LR at different values H

The difference in results is due to the facts that normally distribution has negative values with infinite tails whereas it is not the case for Weibull distribution.

III.3.5 General Case (any type of distributions)

The procedure introduced in figure III.12 is still valid after making the transformation from physical space to the normal space. For the case of any combination of distributions in the limit state function: $G(R,S) = R - S$ .We can make the transformation from physical space into normal space by:

$$u_i = \Phi^{-1}(F_{X_i}(x_i)) \tag{III.22}$$

By introducing this transformation into the limit state expression $G(r,s) = r - s$, we obtain:

$$T(u_R, u_s) = F_R^{-1}(\Phi(u_R); m_R, c_R) - F_S^{-1}(\Phi(u_S); m_S, c_S) \tag{III.23}$$

where $T(.)$ is the image of *G(.)* in the normalized Gaussian space. For this limit state, *k* can be introduced by defining the parametric margin as following:

$$\hat{T}(k, u_R, u_s) = F_R^{-1}(\Phi(u_R); km_S, c_R) - F_S^{-1}(\Phi(u_S); m_S, c_S)$$

Therefore, Lagrange equation is:

$$L(k, u_R, u_S, \lambda) = u_R^2 + u_S^2 + \lambda(km_S - m_S + kc_R m_S u_R - c_S m_S u_S) \tag{III.24}$$

Here the problem in finally to find β_H the safety margin corresponds to the hazard target h_t at the corresponding loading roughness α_s. According to procedure figure III.13, initial value for β_i is assumed and *k* safety factor $\beta_i(k)$ must be calculated and *k* can be obtained by solving the following optimization problem:

$$\begin{aligned} &\min_k : (\beta_i(k)\text{-}\beta_{\mathrm{H}})^2 \\ &\left\| \begin{aligned} &\text{with} : \beta_i(k) = \min_{u_R, u_s} : \quad u_R^2 + u_S^2 \\ &\qquad\qquad \text{under} : \hat{T}(k, u_R, u_S) \le 0 \end{aligned} \right. \end{aligned} \tag{III.25}$$

For the case of linear limit state this is written:

$$\hat{T}(k, u_R, u_S) = km_S - m_S + kc_R m_S u_R - c_S m_S u_S$$

The optimality conditions are:

$$\frac{\partial L(k, u_R, u_S, \lambda)}{\partial u_R} = 2u_R + \lambda k c_R m_S = 0$$

$$\frac{\partial L(k, u_R, u_S, \lambda)}{\partial u_S} = 2u_S - \lambda c_S m_S = 0$$

$$\frac{\partial L(k, u_R, u_S, \lambda)}{\partial \lambda} = km_S - m_S + kc_R m_S u_R - c_S m_S u_S = 0$$

This becomes:

$$u_R(k) = -\frac{1}{2}\lambda k c_R m_S ,$$

$$u_S(k) = \frac{1}{2}\lambda c_S m_S ,$$

$$km_S - m_S = kc_R m_S\left(\frac{1}{2}\lambda k c_R m_S\right) + c_S m_S\left(\frac{1}{2}\lambda c_S m_S\right)$$

$$\lambda(k) = \frac{2(k-1)}{m_S(k^2 c_R^2 + c_S^2)}$$

Then, the problem to be solved is:

$$\min_k \; : \left((u_R(k))^2 + (u_S(k))^2 - \beta_H^2\right)^2$$
$$= \left(\frac{1}{4} m_S^2 \left(\frac{2(k-1)}{m_s(k^2 c_R^2 + c_S^2)}\right)^2 (k^2 c_R^2) - \beta_H{}^2\right)^2$$

The minimum of this quadratic form is zero; we can deduct the safety factor *k:*

$$(k-1)^2 = \beta_H^2 (k^2 c_R^2 + c_S^2)$$
$$k = \frac{2 + \sqrt{4 - 4(1-\beta_H{}^2 c_R^2)(1-\beta_H^2 c_s^2)}}{2(1-\beta_H{}^2 c_R^2)} \qquad \text{(III.26)}$$
$$k = \frac{1 + \beta_H \sqrt{c_R^2 + c_S^2 - c_R^2 c_S^2 \beta_H^2}}{1 - \beta_H^2 c_R^2}$$

III.3.6 Numerical example

Design of a beam in bending

A beam with span of 240 mm and circular cross section is fixed at both ends figureIII.15. At a distance of 80 mm from left end, a force is applied to the beam in the plane y-z, it is normally distributed with mean value equal to $m_S = 705\, N$, and standard deviation equal to $\sigma_S = 185\, N$. The material allowable stress is also normally distributed with mean stress allowable m_{f_y} =80 MPa , the standard deviation of allowable stress σ_{f_y} =15 MPa.

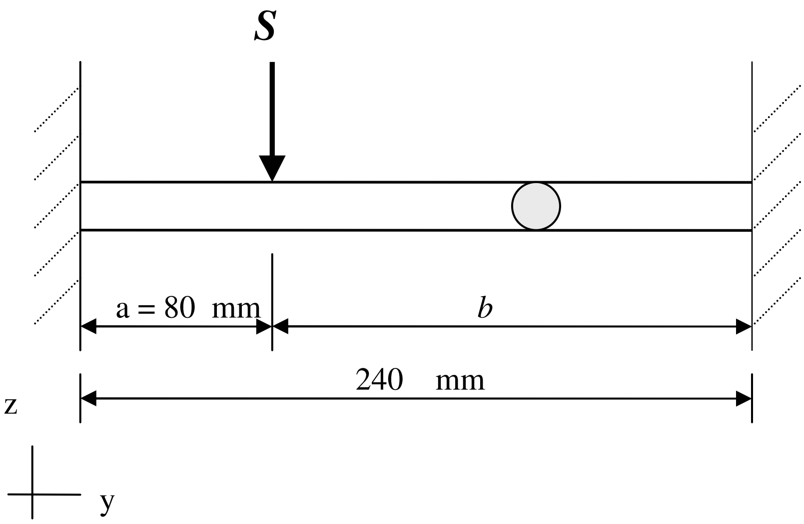


Figure III. 15 Fixed beam with circular cross section

What is the diameter of this beam given that the acceptable level of hazard is 10^{-9} ?

The limit state function is *G* = *R*-*S*, where *R* is the beam capacity in bending and *S* is the applied force. For this loading, the beam capacity in terms of the yield stress is given by: $R = f_y \frac{l^2.W_x}{.a^2.b}$, where *Wx* is the cross-section modulus (for circular section with diameter *d*:

$W_X = \dfrac{\pi.d^3}{32}$), l is the beam length and a and b define the load location. The coefficient of variation of resistance is therefore:

$$\mathrm{cov}_R = \gamma_R = \frac{\sigma_{f_y}}{m_{f_y}} = \frac{15}{80} = 0.188$$

This problem is solved for normal and Weibull distribution with shape parameter 3.44, where the results are produced in table III.2.

Normal distribution	Approximating Weibull
β=4.98	β=3.94
α_s=0.08	α_s=0.3
$m_R = 1.18\times10^4$	$m_R = 3.14\times10^3$

Table III. 2 The result of applying design procedure for the beam problem

The rest of problem is straightforward: knowing the objective mean resistance, the cross-section diameter can be obtained from the relationship corresponding to the mean yield stress.

$$\frac{\mathrm{m_R}.a^2.b}{l^2.W_x} = 80$$

The models of resistance and stress have important effect on the results, approximating normal distribution by Weibull distribution with shape parameter 3.44 is not appropriate, and may lead to inaccurate results.

The precedent calculations have been evaluated at number of load equal to 80, to solve the problem using failure probability as a design target, we calculate the probability of failure equivalent at 80 loads from:

$$P_f = 1 - e^{-h\times n} \tag{III.27}$$

$$P_f = 1 - e^{-10^{-9}\times 80} = 8\times10^{-8}$$

The safety margin required to achieve this value is calculated by extreme value

Failure Probability	Extreme Value (Gumbel)
$\beta = -\Phi^{-1}(P_f) = 5.24$	$\beta = -\Phi^{-1}(P_f) = 5.24$
$m_R = 4.738\times10^4$	$m_R = 7.942\times10^4$
$\alpha_S = 8.979\times10^{-3}$	$\alpha_s = 5.382\times10^{-3}$

Table III. 3 Failure probability and extreme value design

At a constant hazard 10^{-9}, failure probability P_f increases with the application of loads n (equation III.27) and therefore safety index β decreases, we have traced this decrease with the number of load application (figure III.16). The fitted equation is:

$$\beta_{H=10^{-9}}(n) = 6.0387n^{-0.0322}$$

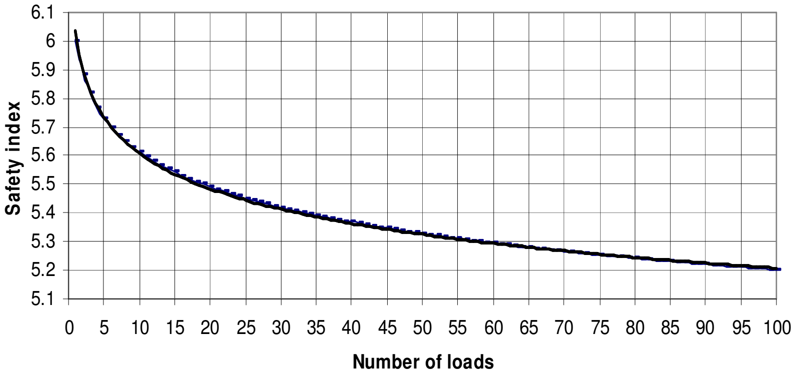


Figure III. 16 Safety margin with number of load application at 10^{-9} hazard

The results illustrates that in the case of repetitive loading to achieve intrinsically reliable design (at hazard level 10^{-9}) failure probability attains this target with mean resistance higher 4 times than hazard approach, and extreme value approach attain it at 6.7 times.

III.4 Hazard based design Optimisation (HBDO)

Design goal of any engineering system is to best fit the performance requirements with minimal cost. For most of products or structural components, a specified level of reliability or safety is implicitly or explicitly imposed as a design target. Clearly, there is a contradiction between cost and reliability, because of the fact that to increase reliability, a cost must be paid either by choosing better materials or by applying a higher quality control, maintenance and testing procedures. Obtaining the cheapest design under reliability constraints is called **R**eliability **B**ased **D**esign **O**ptimisation (RBDO). For the case of repetitive load and non-degraded products, it was shown that hazard based design gives robust design as it is less sensitive to the number of load applications, compared to the failure probability. For this reason, the following subsections aim at providing an alternative approach which is based on hazard as a design target for non-degraded products, called Hazard based design optimisation (HBDO).

III.4.1 RBDO and Life cycle cost

Generally, the expected total cost C_T can be expressed in terms of all costs involved in the structural system, from birth to death. It also includes inspection, maintenance, repair and operating costs [Fra-03], leading to:

$$C_T = C_I + C_F + C_M + C_S + C_R + C_U + C_D \qquad \text{(III.28)}$$

where C_I is the initial construction cost, C_F is the expected failure cost, usually defined as: $C_F = C_f \times P_f$, (C_f being the cost of failure consequences), C_M is the expected preventive maintenance cost, C_S is the expected inspection cost, C_R is the expected repair cost, C_U is the expected use cost and C_D is the expected recycling or destruction cost, which is particularly important for sensitive structures, such as nuclear power plants.

In practice, the design objective of only *minimizing the expected total cost* is not yet applicable, and is somehow dangerous for practical use. For example, if the designer

underestimates the failure consequences with respect to the initial cost, the optimal solution will allow for high failure rates, leading to accept the use of low-reliable structures. The extrapolation to rich and poor countries or cities, leads also to low reliability levels in poor countries (or cities) because of the lower failure costs, as human lives and constructions have statistically lower monetary values. One can imagine the political consequences of such a strategy. At least theoretically, the correct estimation of the failure cost should lead to coherent results. The problem of cost estimation is even more complicated when talking about the whole lifetime management, because the failure cost may change along the structure lifetime due to socio-economical considerations (e.g. life quality of the society). In all cases, special care is strongly required when minimizing the expected total cost, even when other reliability constraints are considered.
Basically, the RBDO aims at minimizing the total expected cost C_T (figure III.14) which is given in terms of initial cost C_I (including design, manufacturing, transport and construction costs) and direct failure cost C_f [Mad-96].

$$\begin{aligned}&\min_{\mathbf{d}} : C_T(\mathbf{d}) = C_I(\mathbf{d}) + C_f \; P_f(\mathbf{d})\\&\text{subject to} : G(\mathbf{d}, X) \leq 0\end{aligned} \qquad \text{(III.29)}$$

where G(.) is the limit state function and the failure probability is given for independent normal variables by:

$$G(\mathbf{d}, \mathbf{X}) = R - S$$

$$P_f(\mathbf{d}) = \Phi(-\beta(\mathbf{d})) \qquad \text{with :} \qquad \beta(\mathbf{d}) = \frac{m_R - m_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

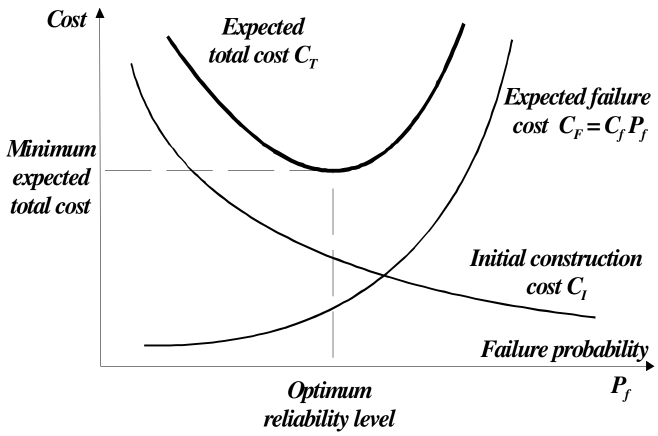


Figure III. 17 Evolution of the costs in function of the failure probability

The total cost equation III.29 indicates that the possible increase of initial cost should be balanced by a decrease in the *risk* C_F (i.e. product: $C_f P_f$) figure III.17. The minimization is carried out for the design parameters such as member sizes, structural configuration and material parameters. These design parameters may correspond to probabilistic distribution parameters: cost is related to the mean value when it represents the nominal design value and to standard deviation when it represents the quality control and the dispersion reduction aspects.

Due to difficulties in estimating the failure cost C_f (especially when dealing with human lives and environmental deterioration, political consequences,…), the direct use of the above equation is not that easy. For design purpose, an alternative to the expect total cost formulation is usually to minimize the initial cost under a prescribed reliability constraint [Mos-77]:

$$\begin{aligned} &\min_{\mathbf{d}} : C_I(\mathbf{d}) \\ &\text{subject to} : P_f(\mathbf{d}) \le P_{ft} \\ &\qquad : \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \end{aligned} \tag{III.30}$$

where $\mathbf{d}^L$ and $\mathbf{d}^U$ are respectively the lower and upper bounds of the design variables and P_{ft} is the admissible failure probability, which is set on the basis of engineering state-of-knowledge and experience. An equivalent formulation is defined in terms of target reliability index β_t:

$$\begin{aligned} &\min_{\mathbf{d}} \ C(\mathbf{d}) \\ &\textit{subject to}: \ \beta(\mathbf{d}) \ge \beta_t \\ &\qquad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U \end{aligned} \tag{III.31}$$

This formulation has the advantage of avoiding the failure cost computation. Nevertheless, the failure consequences can be indirectly included by selecting suitable target safety levels. This problem can be solved using the procedure proposed by Aoues & Chateauneuf [Aou-08].

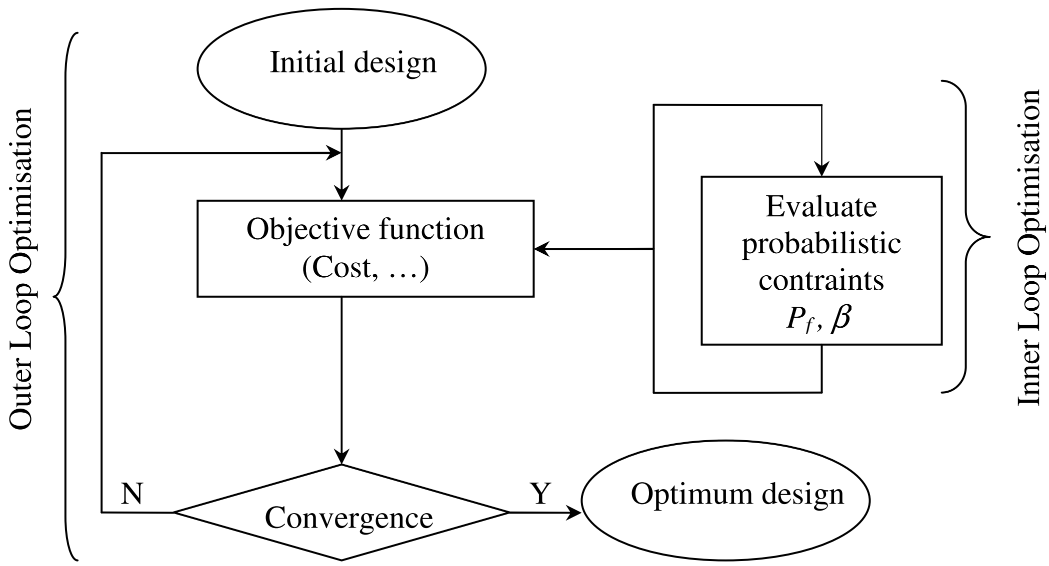


Figure III. 18 Nested RBDO Process

III.4.2 HBDO Hazard Based Design Optimisation

Designing for negligible hazard (Intrinsically reliability) is proposed for the case of repetitive loading and non-degradable components.

Recalling figure III.18, HBDO is different in terms of the inner loop (evaluate probabilistic constraints).Here hazard target is set to the value 10^{-9} to obtain the intrinsically reliable design which can support the repetitive loading given the resistance is not degraded with the application of load.

$$\left\| \begin{array}{l} \min C_I(d) \\ under \quad H \le H^T \end{array} \right. \tag{III.32}$$

Inner loop (evaluate probabilistic constraints)

Limit state function is expressed in normalized space

$$H(U_1, U_2) = \beta + \alpha_R U_1 - \alpha_S U_2$$

First step in to define safety index and loading roughness $\beta(d_k, u_l)$, $\alpha_s(d_k, u_l)$ in terms of d_k, u_l where d_k deterministic dimensions, and u_l are normalized random variables . Second step is to define hazard $h(d_k, n)$ where *n* number of loads in terms of $\beta(d_k, u_l)$, $\alpha_s(d_k, u_l)$ using reliability integral equation III.12. Here we can distinguish between two states:

We suppose a constant value for n (say 80): given that hazard is converged approximately to constant value after few applications of loads. We suppose that n is modelled by discrete random variable such as Poisson distribution, in this case we can calculate n according to the distribution parameter. Then we set hazard equation to the target 10^{-9}, we begin our optimisation loop by giving a starting guess value for d_k. In this case the optimisation algorithm presented in figure III.17 becomes in figure III.18.

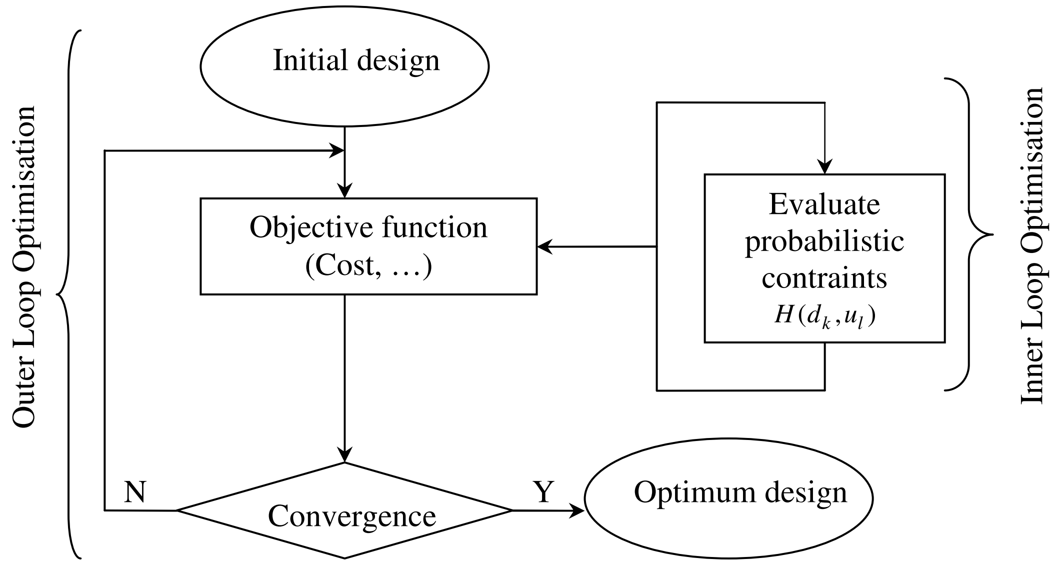


Figure III. 19 HBDO algorithm

- **Optimisation numerical example**

Let us consider the two-bar system shown in figure III.17; the system is supported at two nodes A, B and subjected to a vertical load *P* at node C. The cross-sections are hollow cylindrical with area $S_i = 2\pi . r_i . e_i$ (i=1,2) and moment of inertia $I_i = \pi\, r_i^3 e_i$. The cost per unit weight is given by $c_0 = 1$€ /kg. Suppose the Young's modulus E is normally distributed with mean value $m_E = 210$ GPa, and standard deviation $\sigma_E = 11$ GPa, the applied force is normally distributed with mean value $m_P = 50000$ N, and standard deviation $\sigma_P = 8500$ N .The design criteria are related to member buckling.

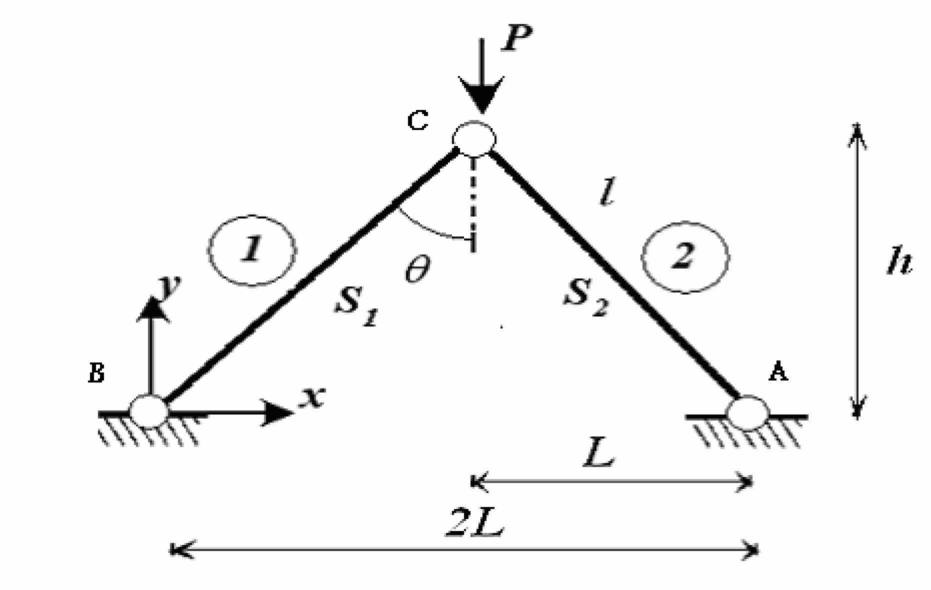


Figure III. 20 Two-link structure under vertical force

The normal force value in each is calculated as:

$$F = \frac{P}{2\cos\theta} = \frac{1}{2}\frac{\sqrt{L^2+h^2}}{h}P$$

The initial cost function is given in terms of material volumes as following

$$C_I = C_0 \rho g V = 4\pi R\, l\, e g C_0 = 4\pi R \mathrm{e} \sqrt{L^2+h^2}\, g C_0$$

The limit state function related to is :

$$G = P_{cr} - P \tag{III.33}$$

where P_{cr} is Euler buckling force, calculated by:

$$P_{cr} = \frac{\pi^2 E I}{l^2} = \frac{\pi^3 E r^3 \mathrm{e}}{\sqrt{L^2+h^2}} = \frac{\pi^3 r^3 \mathrm{e}}{\sqrt{L^2+h^2}} E$$

Substituting the values in equation (III.33) the limit state function becomes,

$$G = a\, E - b\, P$$

with $a = \dfrac{\pi^3 r^3 \mathrm{e}}{\sqrt{L^2+h^2}}$ and $b = \dfrac{1}{2}\dfrac{\sqrt{L^2+h^2}}{h}$.

The corresponding safety index is easily given by:

$$\beta = \frac{a\,m_E - bm_P}{\sqrt{a^2\sigma_E^2 + b\sigma_P^2}}$$

Let us consider that *e* and *L* are constants, and *r* and *h* are the design variables to be optimized. According to the adapted formulation the optimisation is written as following:

1- RBDO

$$\min_{r,h} c_0 V$$
$$under \beta \geq 3.72$$

3- HBDO

we set the hazard to the value 10^{-9} to obtain the intrinsically reliable design assuming that the structure is subjected to repetitive loading.

$$\min_{r,h} c_0 V$$
$$under H \leq 10^{-9}$$

The solutions are performed using MathCAD software. The results obtained are presented in table III.4,

RBDO *e = 3 mm, L=0.5 m*	*HBDO* *e = 3 mm, L=0.5 m*
$r_{op} = 0.012$ m	$r_{op} = 0.012$ m
$h_{op} = 0.223$ m	$h_{op} = 0.226$ m
$V = 1.211 \times 10^{-4}$ m^3	$V = 1.262 \times 10^{-4}$ m^3

Table III. 4 RBDO, HBDO optimization results

In first left column RBDO are performed for fixed values of *r* radius and *L* distance, the result was trivial because the solution gives us a circular hollow section with a thickness equal to 2 microns. Therefore, we have solved the optimization problem for fixed thickness *e* and distance *L*. The radius optimized value was identical whereas the optimal height *h* was grater for repetitive loads (the case of hazard based design) this increase in justifiable because we design for intrinsically reliable target under repetitive load.

III.5 Discussion and Conclusions

In this chapter, we have investigated the case of repetitive load applications in the frame work of probabilistic design. The hazard is considered as a design basis due to its capability to deal with repetitive loads which is far from being than the case of failure probability. Starting from the fact that the load effect has the major importance along the useful period of product life cycle, a description and categorization have been presented. Given, that in resistance time independent products hazard is converged approximately to a constant value after few applications of load. This idea was justified mathematically, graphically and numerically. It was shown that hazard is less sensitive to the number of load application compared with failure probability. Thus, in the case of repetitive loading considering hazard as a design target is more rational than any other target for this case. Then a hazard based design approach is proposed using the normal lognormal and Weibull distributions to model both of stress and resistance and for general case of any different types of distributions. Weibull distribution is presented as a practical and more realistic model can replaced normal distribution in the case of shape parameter 3.44.This approximation is not accurate and results using Weibull distribution were extremely far from normal distribution. Resistance obtained from extreme value solution and single load application was significantly greater than resistance obtained from hazard approach at negligible hazard.

HBDO for hazard target 10^{-9} was compared with RBDO for safety index target equal to 3.72 through an example, the results were logical and the difference between the two costs was not significant, given that in HBDO we design for intrinsically reliable component under repetitive loads, whereas in RBDO we design for single load application with failure probability 10^{-4}. Therefore, HBDO approach has demonstrated that it is robust and realistic for this kind of problems. This method is introduced to design for repetitive load unlike other approaches of structural reliability, which used a failure probability as a design target and a single load application. Weak components can be get rid of by quality control measures (i.e. proof testing for mechanical component, burn in for electrical components). However, in single load approaches, if the design is able to survive the first load, it will survive. For certain design applications like space shuttle or aeroplane; it is more reasonably to design for repetitive load than single load. Finally, in our opinion this method is a good design tool if the statistical data used is accurate, and if the software evaluating reliability integral is a robust package.

Chapter IV. Reliability of degraded structures

IV.1 Introduction

The previous chapters deal with the first two phases of product life, this chapter is concerned with the wear-out phase in order to allow for full description of the life cycle design basis.
The first part of this chapter describes statistical degradation approaches, followed by the physical approaches including the probabilistic degradation models in the cases of instantaneous and cumulated margins.

IV.2 Degradation Modeling

Modelling the degradation is necessary to establish reliable models describe system behaviour along the life-cycle, considering the operational conditions. It is necessary to consider separately various components, on one hand, and to take into account the interactions between various subsets of components under the operational environment, on the other hand. This requires coupling the physical-chemical mechanical behaviour, in order to quantify the evolutions of the properties during time.

Modelling the degradation mechanisms implies the use of mathematical tools allowing for good representation of the time-dependent evolution. The reliability analysis has to be updated by models having the ability to consider the knowledge and the experience feedback. The main mechanical degradation mechanisms can be listed in table IV.1.

According to structural reliability theory based on stress-resistance model, decreasing resistance characterizes degradation. In reliability theory, degradation is often treated as a "first passage "or "barrier crossing" problem (figure IV.1) [Mad-86].

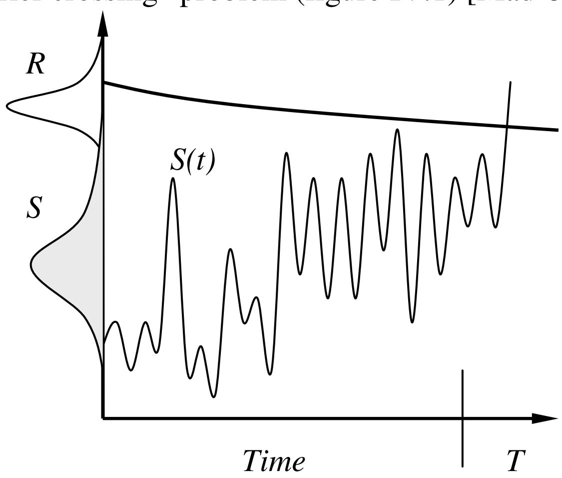


Figure IV. 1 Evaluation of Resistance and stress samples

In the following subsections, we introduce how degradation can be modelled according to stress-resistance interference model.

Degradation mechanisms	Types of degradation mechanisms
Erosion (Mechanical action)	(i) abrasion : sliding action, rolling action (ii) rubbing : sliding action, rolling action
Corrosion (chemical or electro-chemical action)	(i) solid/solid (ii) solid/liquid (iii) solid/gaseous (iv) mono-solid (v) stress corrosion surface protection breakdown by wear process
Fatigue	(i) low cycle fatigue (ii) High cycle fatigue (iii) Thermal fatigue (iv) Corrosion fatigue
Surface degradation	(i) fretting (ii) pitting (iii) spalling (iv) cavitation action
De-fastening	(i) from vibration (ii) from repeated shocks (iii) from thermal cycling
Creep	(i) at normal temperature (ii) at high temperature
Ageing	(i) Thermal (ii) Chemical (iii) Structural (iv) environmental
Fouling	(i) From dirt etc. (ii) By clogging (iii) From wear debris accumulation
Contamination	(i) of liquids (ii) of gases
Leaking	(i) past solid joints (ii) Through solids (permeability)
Thermal	(i) overheating (ii) burning (iii) distortion

Table IV. 1 Main mechanical degradation mechanisms [Car-86]

IV.3 Statistical approaches for degradation

Kapur et al. [Kap-77] have developed stress-resistance models for time dependent resistance and repeated stress applications. Recalling stress-resistance classification of design problems in chapter III (figure III.I) our concerned problem is now depicted in grey blocs in figure IV.2.

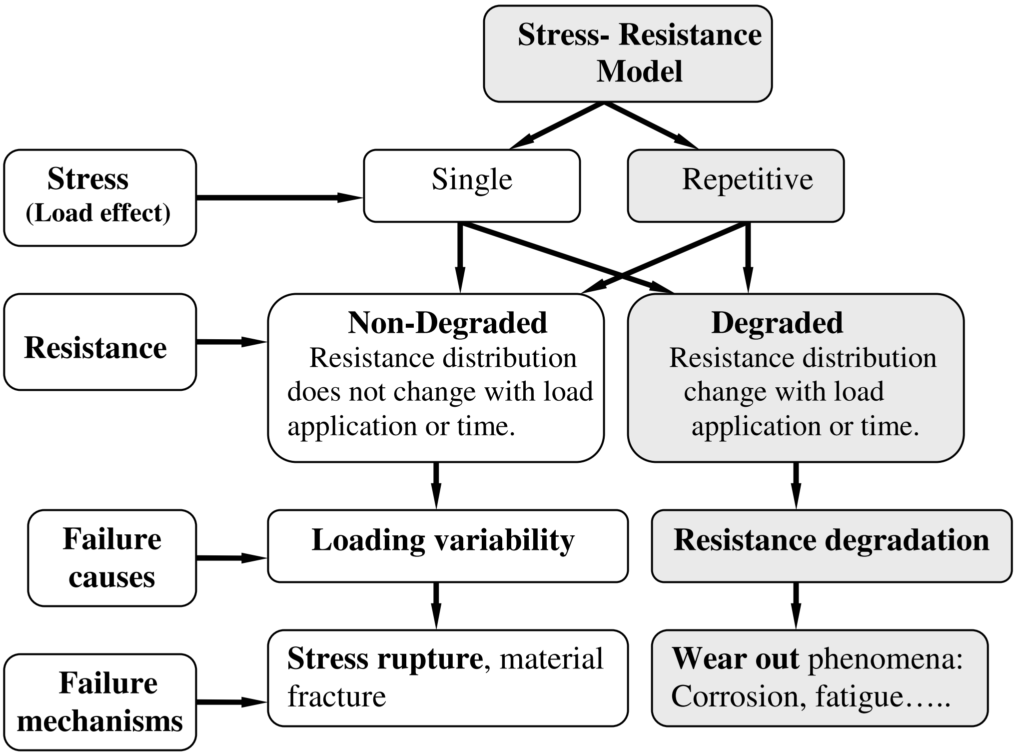


Figure IV. 2 Stress-resistance classification of design problem

In their approach, stresses have been classified into three types with respect to time: *constant*, *cyclic* and *random*. The first two patterns are generally found in laboratory testing whereas, the later corresponds to real-world applications, such as loads produced in vehicle's suspension components by random irregularity of road surfaces. Basically, stress and resistance are classified in three categories, ***deterministic***, ***random-fix*** and ***random-independent*** (figure IV.3). To explain the later two terms, ***random*** means that the uncertainty of stress or resistance at any instant of time or cycle and ***fix*** or ***independent*** refer to the behaviour of the variables with respect to time. Thus, ***fix*** means that the variation with time is given in a fixed manner (i.e. deterministic function…), while ***independent*** means that the successive values of the variables are statistically independent, and thus one value does not give information about the size of subsequent values.

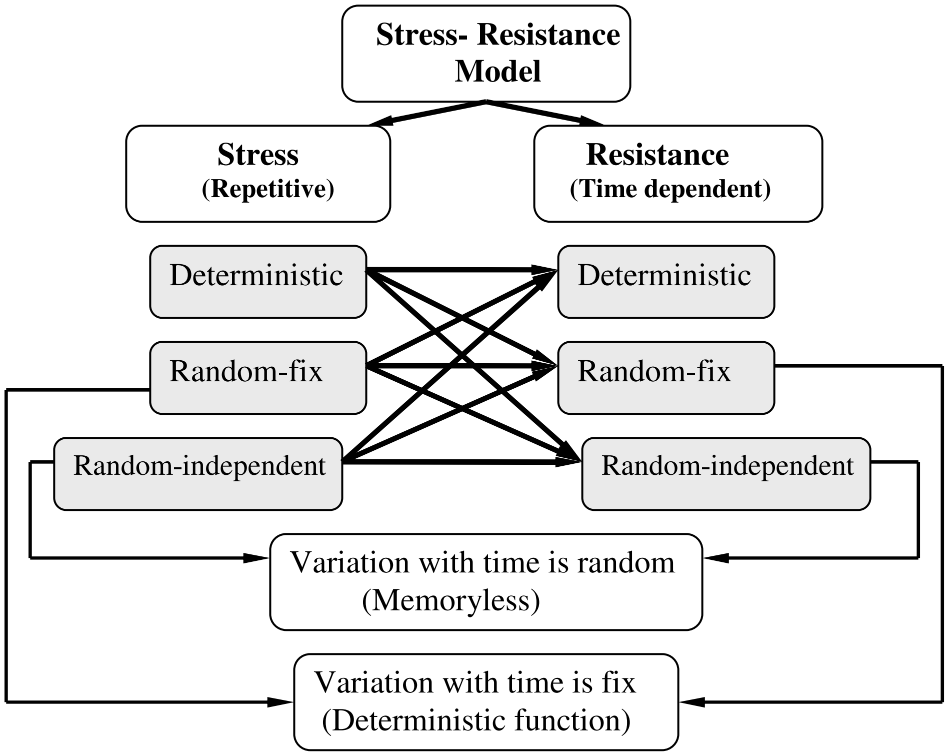


Figure IV. 3 Kapur and Lamberson classification for time dependent stress-resistance model

Kapur et al. [Kap-77] have developed the reliability expression after n applications of loads for the nine possible cases, where, the resistance depends on the number of load applications, their magnitudes and time durations. If the resistance varies only with physical time, the effect is called **aging** (corrosion is an example of aging). If the resistance is a function of the number of load occurrences, the effect is called **cyclic damage**. If it is a function of load occurrences as well as their magnitude, the effect is called **cumulative damage** (**fatigue** is an example for the later two cases). The results of these analyses are summarised in table IV. 2.

Table IV. 2 Kapur and Lamberson time dependent reliability calculation

N°	S	R	Reliability computation for deterministic cycle times	Reliability computation for random cycle times (Poisson's)
1	**Deterministic**	**Deterministic**	$\Re_n = \begin{cases} 0 \text{ if } x_i > y_i \text{ for some } i, & 1 \le i \le n \\ 1 \text{ if } x_i \le y_i \text{ for all } i, & 1 \le i \le n \end{cases}$ where, x_i *and* y_i are the values of stress and resistance for the i^{th} cycle. $\Re_n$, is the reliability after *n* cycles.	$\Re(t) = \sum_{i=0}^{\infty} \pi_i(t)\Re_i$ where $\pi_i(t)$ is the probability of *i* cycles occurring in time interval.(Poisson distributed) $\pi_k(t) = P(N_t = k) = \frac{e^{-\alpha t}(\alpha t)^k}{k!}$ α is the mean occurrence per unit time.
2	**Deterministic**	**Random-fix**	$\Re_n = P(x_n \le y_n) = p(x_0 \le y_0 - a_n) = p(y_0 \le x_0 + a_n)$ $\Re_n = \int_{x_0+a_n}^{\infty} g_0(y_0)dy_0$, where x_0 is the constant stress, y_i is the resistance at the i^{th} cycle given by $y_i = y_0 - a_i$, $i = 1,2,\ldots.$	$\Re(t) = \Re + (1-\Re)e^{-\alpha t}$ where $\Re = \int_{x_0}^{\infty} g_0(y_0)dy_0$ is the reliability for one stress cycle. Here a_i*'s* assumed to be zero.
3	**Deterministic**	**Random-independent**	$\Re_n = \left\{ \int_{x_0}^{\infty} g(y)dy \right\}^n$ where x_0 is the constant stress, $g_i(y)$ is the pdf of resistance y_i during the cycle *i,* if $g_i(y)$ unchanged over time $g_1(y) = g_2(y) = \ldots = g_n(y) = g(y)$	$\Re(t) = e^{-\alpha t(1-R)}$ Where, $\Re = \int_{x_0}^{\infty} g(y)dy$ is the reliability for one stress cycle.
4	**Random-fix**	**Deterministic**	$\Re_n = P(x_n \le y_n) = p(x_0 + b_n \le y_0) = p(x_{0^2} \le y_0 - b_n)$ $\Re_n = \int_0^{y_0-b_n} f_0(x_0)dx_0$ where $x_i = x_0 + b_i$ and x_i, is the stress in the i^{th} cycle, b_i are known non-negative constants and y_0 is the constant resistance. The PDF of x_0 is $f_0(x_0)$.	$\Re(t) = \Re + (1-\Re)e^{-\alpha t}$ where $\Re = \int_0^{y_0} f_0(x_0)dx_0$ is the reliability for one stress cycle. Here, x_0 is random-fix stress with known PDF $f_0(x_0)$ that does not vary with time, y_0 is deterministic constant resistance.

Table IV.2 continue

N°	*S*	*R*	*Reliability computation for deterministic cycle times*	*Reliability computation for random cycle times (Poisson's)*
5	**Random-fix**	**Random-fix**	$\Re_n = P(x_n \leq y_n) = p(x_0 + b_n \leq y_0 - a_n) =$ $p(x_0 \leq y_0 - a_n - b_n)$ $$\Re_n = \int_0^\infty g_0(y_0)\left(\int_0^{y_0 - a_n - b_n} f_0(x_0)dx_0\right)dy_0$$ Stress is given by $x_i = x_0 + b_i$ and the resistance is given by $y_i = y_0 - a_i$	$\Re(t) = \Re + (1-\Re)e^{-\alpha t}$ where $$\Re_i = \int_0^\infty g_0(y_0)\left(\int_0^{y_0} f_0(x_0)dx_0\right)dy_0 = \Re$$ is the reliability for one stress cycle. Here, x_0 is random-fix stress with known PDF $f_0(x_0)$ that does not vary with time, and y_0 is deterministic constant resistance.
6	**Random-fix**	**Random-independent**	$\Re_n = P(y_{\min} > x)$ $$\Re_n = \int_0^\infty f(x)[1 - G(x)]^n dx$$ *f(x)*, is PDF of stress the random resistance are independent identically distributed, with PDF g(y),G(x) is CDF of resistance.	$$\Re(t) = \sum_{i=0}^{\infty} \pi_i(t)\Re_i$$ $$\Re(t) = \sum_{i=0}^{\infty} \frac{e^{-\alpha t}(\alpha t)^i}{i!}\int_0^\infty f(x)\left(\int_x^\infty g(y)dy\right)^i d$$ $$\Re(t) = \int_0^\infty f(x)e^{-\alpha t G(x)}dx$$
7	**Random-independent**	**Deterministic**	$$\Re_n = \left\{\int_0^{y_0} f(x)dx\right\}^n$$ where y_0 is the constant resistance, *f(x)* the PDF of stress assuming that $f_1(x) = f_2(x) = \ldots = f_n(x) = f(x)$	By reciprocity with case 3, we get $$\Re(t) = e^{-\alpha t(1-R)}$$ where $\Re = \int_0^{y_0} f(x)dx$ is the reliability for one stress cycle.
8	**Random-independent**	**Random-fix**	$\Re_n = P[\max x < y]$ $$\Re_n = \int_0^\infty g(y)[F(y)]^n dy$$ Stresses are each load application *i.i.d* with pdf *f(x)* and CDF *F(x)*.	By reciprocity with case 6, we get $$\Re(t) = \int_0^\infty g(y)e^{-\alpha t(1-F(y))}dy$$
9	**Random-independent**	**Random-independent**	$$\Re_i = \int_0^\infty f_i(x)\int_x^\infty g_i(y)dydx$$ $$\Re_n = \prod_{i=1}^{n} \Re_i$$ where, R_i is the reliability at the i^{th} cycle.	$\Re(t) = e^{-\alpha t(1-\Re)}$ where $\Re = \int_0^\infty f(x)\int_x^\infty g(y)dydx$ is the reliability for one stress cycle.

Smith et al. [Smi-94] tackled the resistance deterioration behaviour by assuming that the resistance distribution is bimodal (figure IV.4). They divided the product into "weak" and "strong" components and the resistance decays according to physical model, specifically **Paris crack propagation law**, because of the fact that fatigue failures account for a large percentage of observed failures. Here, the initial resistance distribution $f_R(r_0)$ is the probability density function of the population resistance, given by:

$$f_T(r_0) = pf_w(r_0) + (1-p)f_m(r_0)$$

where $f_w(.)$ and $f_m(.)$ are the initial probability density functions of the weak and the main populations respectively and $\boldsymbol{p}$ is the fraction of weak components.
For fatigue crack propagation, the authors presented the results in terms of hazard function graphs (figure IV.5) for different cases, by changing the applied load the weak to main population rates and Paris law characteristics parameters.

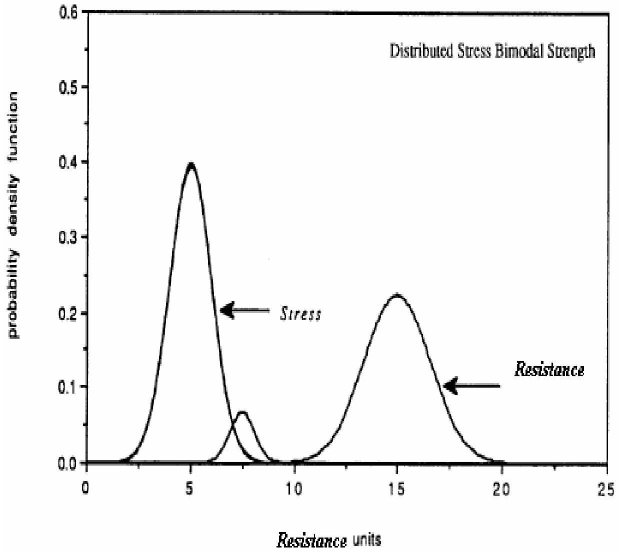


Figure IV. 4 Distributed stress bimodal resistance [Smit-94]

According to Smith et al. [Smi-94] a reliability model is derived from the assumption of resistance decay, governed by the quality of the population and existing physical laws. This degradation model provides a more credible and consistent explanation of the three phase failure life. Components are continually in a process of wear-out and failure results as component deteriorates into the stress region. At any time, components are memory-less and thus subjecting the entire population to overstressing will affect even the strongest components, where the degree of damage is governed by the magnitude of stress. Quality plays an important role in reliability deterioration model. The authors indicated that these concepts may help in preventive maintenance.

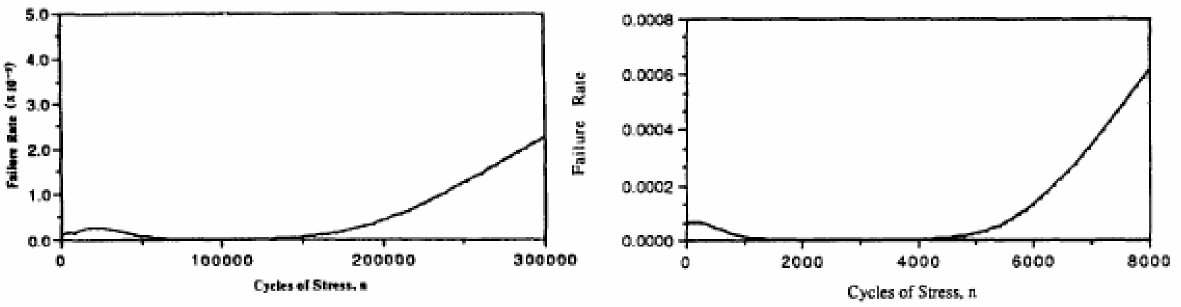


Figure IV. 5 hazard versus number of cycles two cases [Smi-94]

Place et al. [Plac-99] proposed a model based on stress-resistance interference model and damage accumulation. The approach consists in modelling the growth in damage as function of time and system loading, using appropriate damage accumulation parameters. Failure occurs when the damage exceeds the limit of damage tolerance. The generalised damage accumulation model includes linear and non linear growth rate as

$$\frac{d}{dN}\left(\frac{D}{D_0}\right) = \alpha_i \left(\frac{D}{D_0}\right)^q$$

where D is the damage after N cycles of constant stress amplitude, D_0 is the initial damage, α_i is the incremental damage factor and q is the factor describing the rate of damage growth over Δn_i. They applied their approach on helicopter gearbox components (bearings, seals, shafts, casing and lubricating oil) and came up with curves representing the relationship between the failure probability and the operating time.

This application concerned the fatigue for gear teeth, bearings and shafts. The fatigue model can be adapted to the widely used Miner's law or Paris laws. Such a model aims at simulating the damage accumulation process and can be validated using existing *S–N* data. Damage accumulation model is then developed for wear and corrosion processes. Stress-resistance interference models are then used to obtain a cumulative probability function. In their study, the operating environment of the gearbox is characterized as a series of operating states; each of them has its own damage accumulation parameters. Though the knowledge of the helicopter's operating regime, a risk analysis can be performed to assess the risk of being in a particular state, with the associated consequences. Such an analysis can be used to assess the probability of failure against time (or flying hours), to give a quantitative value for the system reliability.

IV.3.1 Carter wear out design approach

[Carter-97] represented the degradation in terms of component age (figure IV.6). In this approach, full representation of wear is obtained by modifying the interference model to include the damage process *erosion, corrosion, creep, fatigue…..etc*. Wear process can be modelled by the equation

$$f_R(.) = \text{function}\,[f_{R_0}(.), f_S(.), n]$$

where $f_{R_0}(.)$ is the initial distribution of resistance, $f_S(.)$ is the load distribution and n is the number of load applications. Wear process is represented by damage threshold distribution (figure IV.7. a), in which if the value of the stress goes above, the damage is set in; and below no damage is done (figure IV.7.a).

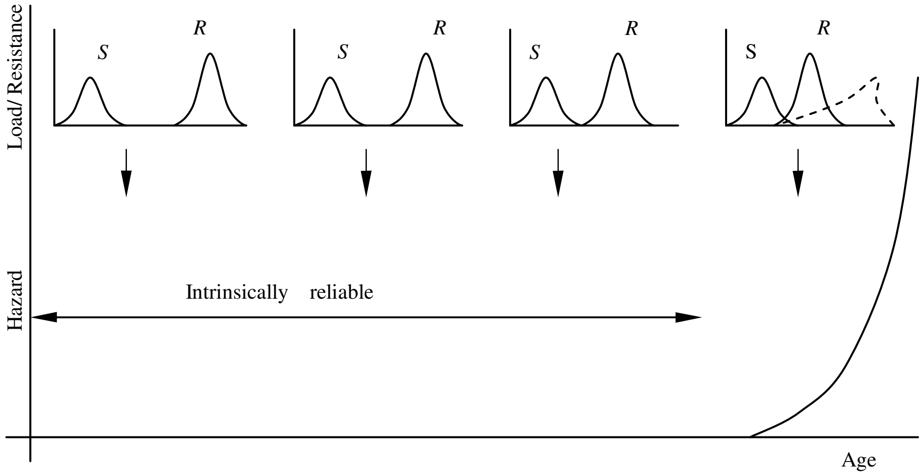


Figure IV. 6 Statistical representation of wear process [Car-97]

For the particular case of fatigue, the damage threshold is known as the endurance or fatigue limit. The damage is a function of the interference between any load and the threshold of the material damage. Naturally, the damage function is different for every wear process. In the case of fatigue, the damage resistance function is represented by the well known S-N curve (figure IV.7.b).

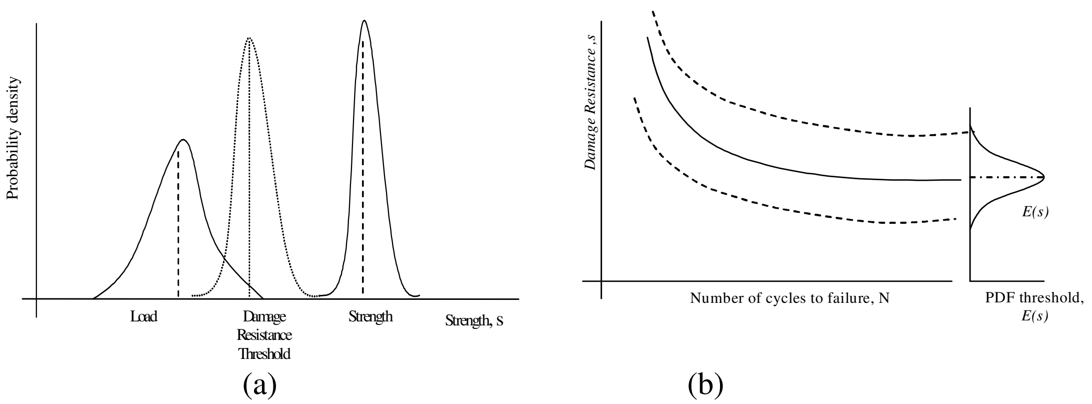


Figure IV. 7 (a) load – resistance interferences for wear (b) Probabilistic S-N damage curve

The uncertainty of damage resistance like any other property is statistically distributed (figure IV.7.a). In this approach, the design for the random phase must be achieved, according to the concept of intrinsically reliable discussed in chapter III. This considers as a mandatory condition that the component has survived in the second part of life, otherwise, any failure that occurs will be due to stress rupture and not attributable to wear; i.e. it will be assumed that intrinsic reliability is achieved.

IV.3.1.1 Wear out life distribution and hazard

To determine quantitatively the distribution of life, transformation function (damage threshold into life distribution) must be evaluated: i.e. the value of *N,* corresponding to each endurance limit *E,* has to be calculated. For this purpose it will be assumed that Miner's rule can be applied to each $s-N_F$ (stress-number of loads at a certain probability of failure) curve in order to estimate individual lives. In the case of fatigue, the population is assumed to be well finished and homogeneous. Any other necessary condition must be satisfied for other wear processes.

If the median curve is given by:

$$N = \xi(s) \tag{IV.1}$$

The set of $s - N$ curves is shown in the following equation:

$$N = \frac{\text{Constant}}{S^m} \tag{IV.2}$$

which can be written in a generalized form as:

$$N = \xi(s - z) \tag{IV.3}$$

where z is defined by $z = s_F - s_{50}$ in figure IV.8 (s_{s_0} being the stress at 50% of lifetime probability)

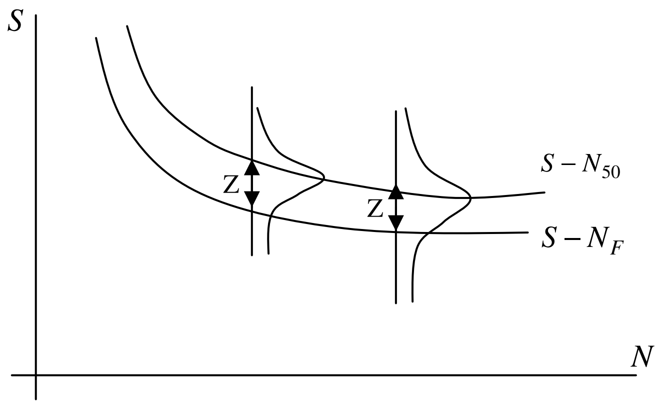


Figure IV. 8 S-N curve at F % failure probability

Let us consider the threshold resistance is E where

$$E = \overline{E} + z \tag{IV.4}$$

The damage done by the application of one load giving a stress of magnitude S_i is obtained from the damage law as:

$$\Delta d_i = \frac{1}{\xi(s_i - z)}$$

The number of times the load will be applied is:

$$n_i = nS(s_i)ds$$

where *n* is the total number of load applications. Hence, the damage inflicted by the particular load is given by:

$$d_i = n_i \Delta d_i = nS(s_i)ds \frac{1}{\xi(s_i - z)} \tag{IV.3}$$

The total damage resulted by all the loads from the distribution $S(s)$ on an item of threshold damage resistance E is then the sum of the damage done by all the individual loads in the distribution $S(s)$, i.e.

$$d = \sum d_i = \int_0^\infty \frac{nS(s)}{\xi(s-z)} ds \qquad \text{(IV.4)}$$

In equation (IV.4) *s* takes all the values in the distribution *S(s)*. By definition, *d =1* at failure, therefore:

$$\sum d_i = \int_0^\infty \frac{NL(s)}{\xi(s-z)} ds = 1 \qquad \text{(IV.5)}$$

where N is the number of cycles to failure.

$$N = \frac{1}{\int_0^\infty \frac{S(s)}{\xi(s-z)} ds} \qquad \text{(IV.6)}$$

Thus, for each value of *E* given by $E = \overline{E} + z$, the time to failure can be calculated. The relationship between *E* and *N* is one to one. It follows that the number having threshold damage resistance *E* will be the number of items failing at *N*, or mathematically

$$E(s)ds = f(N)dN$$

where $f(N)$ is the probability density function of life failure:

$$f(N) = E(s)\frac{dE}{dN} \qquad \text{(IV.7)}$$

The cumulative failures are given by:

$$F(N) = \int_0^N f(N)dN$$

and the hazard by:

$$h(N) = \frac{f(N)}{1 - F(N)} \qquad \text{(IV.8)}$$

Repeating for all values of *E*, the wear life pattern is completely solved for any wear process, given the S-N distribution.

IV.4 Structural physical approaches [Cha-08]

The degradation of a system (or constitutive material) is the effect of a slow and irreversible evolution of one or more properties starting from an initial point, generally taken as the end of the fabrication cycle. Ageing becomes a problem when it corresponds to a deterioration of the properties affecting the operation performances (aspect, mechanical resistance, drift of functional performances) and the properties affecting safety (electric insulation, gas leak or liquid, toxicity…). The behaviour with ageing requires the identification of the loads in service (distribution of the extreme loads, impact of the environmental conditions …) that are applied to the system and the study of their incidence over the life cycle.

IV.4.1 Safety Margin

To formulate the problem of reliability in a universal way based on the concept of the safety margin, two principal entities are defined as follows:

- **The Resistance *R (T),*** which represents the resistance of the equipment to the stresses (mechanical, thermal…). The admissible threshold of an observable or un-observable effect, such as displacement, crack length and width, critical damage, time to failure…

- **The Stress *S (T),*** which represents the applied environment: force, pressure, temperature…etc. The effects of the applied environment: mechanical stresses, temperature, internal displacement between components…etc. The accumulated magnitude resulting from the applied stresses during the equipment age: fatigue damage, cracking, creep…

Generally, the margin can be described by the difference between the resistance and the stress:

$$G(r,s,t) = R(t) - S(t) \qquad \text{(IV.9)}$$

where *R(t)* is the available resistance in the system and *S(t)* is the stress; the two entities are time dependent.

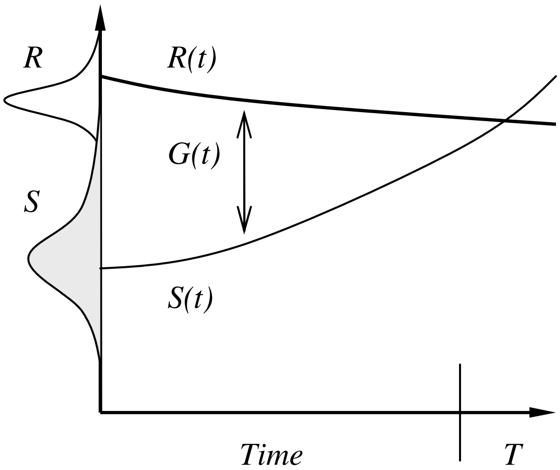


Figure IV. 9 Evolution of the resistance and the stress

The degradation can thus be interpreted as being the reduction of the margin; it implies implicitly (the degradation against one or more failures modes). Two types of cases are distinguished:

- ***Instantaneous margin***: it concerns to the situation where the resistance *R(t)* decreases with time (calendar or operational) accompanied or not by the increase in the stress *S(t),* which can be also generated by degradations. According to Kapur et al. [Kap-77] this classified as random-fix, i.e. the resistance decays with time independently of load application.

- ***Cumulated margin:*** it is related to the difference between the acceptable or operational threshold and the cumulated degradation (such as the propagation of crack by fatigue). In this case, the effect of the environment *S(t)* increases with operation time until the consumption of the available resource *R(t).*This case is classified as random-independent by Kapur et al. [Kap-77].

IV.4.2 Degradation model

For a degradable component, various models can be proposed according to the studied phenomenon. A general model consists in dividing the lifespan into four phases: phase of initiation (or incubation), starting phase, propagation phase and acceleration phase (figure IV.10). This curve resembles the hazard function curve in the random phase corresponds to initiation part then starting, propagation and acceleration phases correspond to wears out phases.

- **Initiation Phase (or incubation):** during this phase, the degradation mechanisms do not have an effect on the system, because of protection measures. This phase is more or less large, according to the degradation mechanism. In the case of corrosion, it may vary from a few days for the steel in a salt vapour, to several decades for protected steels.

- **Starting Phase:** In this period, the aggressive factors act directly on the system when followed by protection loss with or without the increase in the stress level. This phase is usually small compared to the structures lifespan.

- **Propagation Phase:** in this phase the system degradation is slow and often continuous, generating increasing damage. The system continues to perform properly, despite its deviation from the nominal conditions (or initial). Generally, this phase is accompanied by the presence of a significant defect and mostly detectable.

- **Acceleration Phase**: in this phase, the defect becomes so important that it contributes significantly to the acceleration of the degradation process. In other words, acceleration results from the interaction between the defect and the environment, and not only from the environment. In this phase, it is often too late to perform in the normal operations of maintenance.

According to the considered mechanism, some of these phases can be very short, and are consequently neglected in the model. Two simplified models are often given in the literature, as indicated in figure IV.11. The multi-linear model consists of two principal phases: initiation and propagation, followed by acceleration phase (often neglected in the study of the useful life duration). The non-linear model is represented by a curve of continuous degradation in time, when degradation is continuous and progressive with an increasing rate.

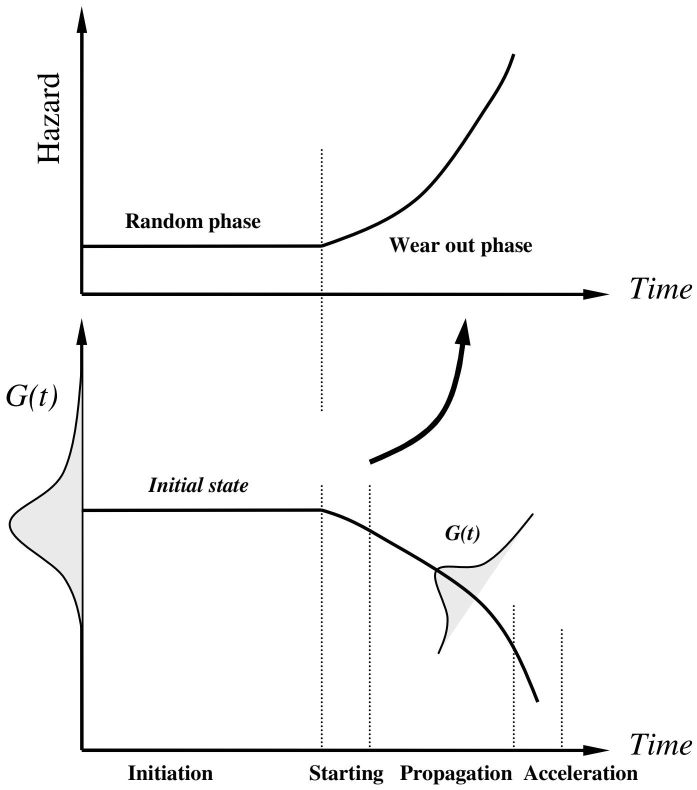


Figure IV. 10 Margin of a degraded component and hazard

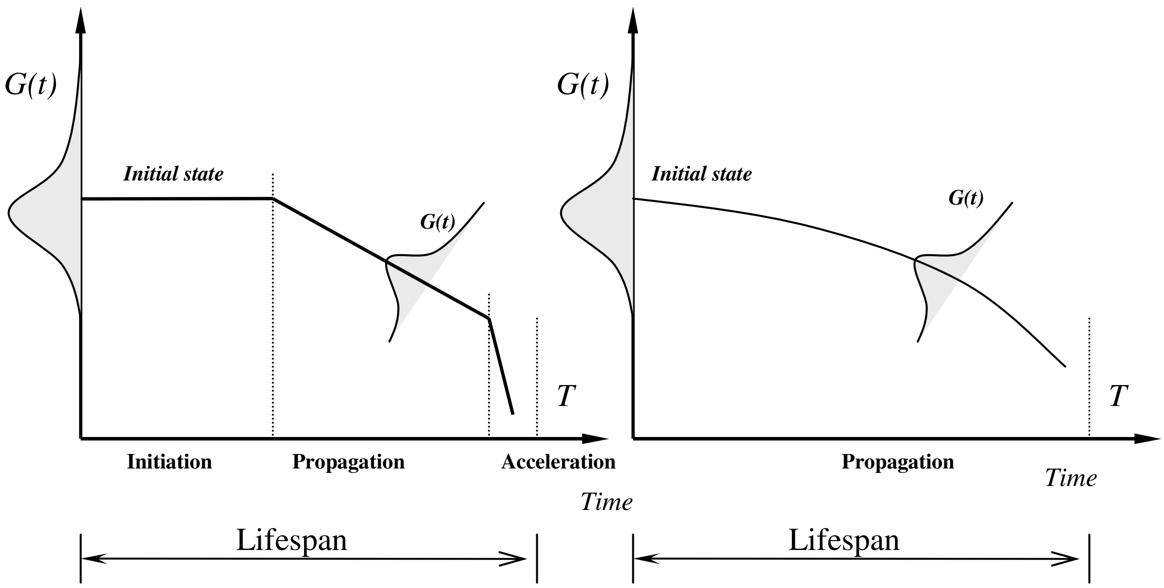


Figure IV. 11 simplified degradation Models

It is considered in the following the two situations indicated above, degradation of the instantaneous margin and the degradation of the cumulated margin.

IV.4.2.1 Instantaneous margin degradation

This situation corresponds to the reduction in the resource *R (t)* with age (or service time). This reduction is related to the history of the environment (e.g. evolution of loading and temperature). It can be accompanied (or not) by the increase in the effect of the environment, as for example the case of thickness loss by corrosion which generates an increase in the stresses under the same loading.

The probabilistic modelling of degradation requires considering the uncertainty evolution with time. This uncertainty results from two causes:

1) Inherently degradation in the system according to the operational conditions,
2) The imperfection in the operational conditions, the system state and the degradation models.

The parametric model of degradation can be simplified by assuming that degradation during time is described by the product of initial resistance (random) by a deterministic degradation function:

$$R(t) = R_0 f(t) \tag{IV.10}$$

where *R(t)* is resistance during time, R_0 is resistance at the initial state (i.e. *t=0*) and *f(t)* is degradation as a function of the component age. This model is identical to Kaptur's model [Kap-77] resistance random fix.

For the case where the function of degradation is independent of the loading history, several authors, such as Mori et al [MOR-01], proposed the following form:

$$f(t) = 1 - a\,t^b \tag{IV.11}$$

where, ***a*** indicates the rate of degradation; the ***b*** is the nature of degradation

Degradation Form	Expression	Example
Linear degradation	$f(t) = 1 - a\,t$	Corrosion, wear
Parabolic degradation	$f(t) = 1 - a\,t^2$	Sulphate attacks
Square root degradation	$f(t) = 1 - a\sqrt{t}$	Controlled diffusion

This model has the advantage of its simplicity, since it only depends on modelling by random variables. However, the influence of the degradation function is not only limited to the mean, but it also affects all the probabilistic distribution of resistance (figure IV.12); i.e. the function *f(t)* also modifies the resistance dispersion, which could be interpreted as a constant coefficient of variation throughout the lifespan or constant loading roughness according to Carter [Car-86,97]. This assumption is not realistic in almost all applications, since the standard deviation of the expectation cannot decrease by degradation.

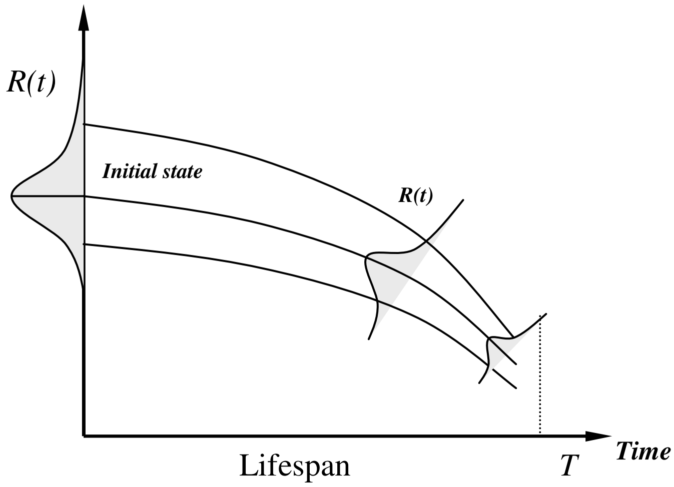


Figure IV. 12 Deterministic model of degradation

A better representation of the resistance evolution consists in defining the degradation functions, not for the random variable itself, but also for its statistical parameters: mean m_R and standard deviation $\sigma_{R.}$

$$\begin{aligned} m_R(t) &= m_{R_0}\ f_1(t) \\ \sigma_R(t) &= \sigma_{R_0}\ f_2(t) \end{aligned} \tag{IV.12}$$

where $f_1(t)$ and $f_2(t)$ are the degradation functions for the mean and standard deviation respectively, and m_{R_0} , σ_{R_0} are the mean and standard deviation at the initial state. This model has the advantage of being able to freely modify the mean and dispersion along with the component age (figure IV.13). However, the initial conditions remain determined for all the system history.

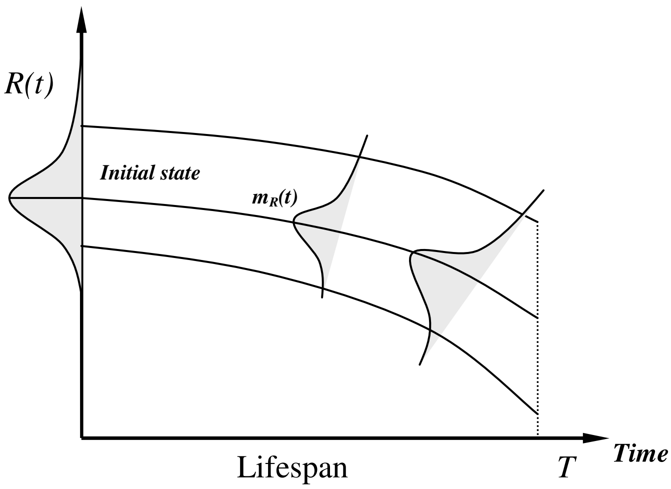


Figure IV. 13 Deterministic model of the degradation parameters

As an example, the following scenarios can be simulated:

• Constant standard deviation: $f_2(t) = \alpha$

• Constant coefficient of variation: $f_2(t) = \alpha\, f_1(t)$

• Standard deviation increases: (linear) $f_2(t) = \alpha\, t$ or (exponential) $f_2(t) = e^{\alpha\, t}$

• The increase in the standard deviation proportional to the reduction in mean resistance:

$$f_2(t) = \frac{\alpha}{f_1(t)}$$

A more realistic framework can be defined by including random events during the life-span; resistance can be more correctly modelled by stochastic processes.

IV.4.2.2 Cumulated margin degradation

This situation corresponds to the cumulated damage until reaching the allowable value for system operation. In this case, the resource corresponds to the threshold (i.e. limit or acceptable value) and the effect of the environment corresponds to the cumulated damage. This accumulation is generated by the operating conditions: mechanical, thermal environment, etc. Therefore, there is coupling between the scenario of loading and the degradation.

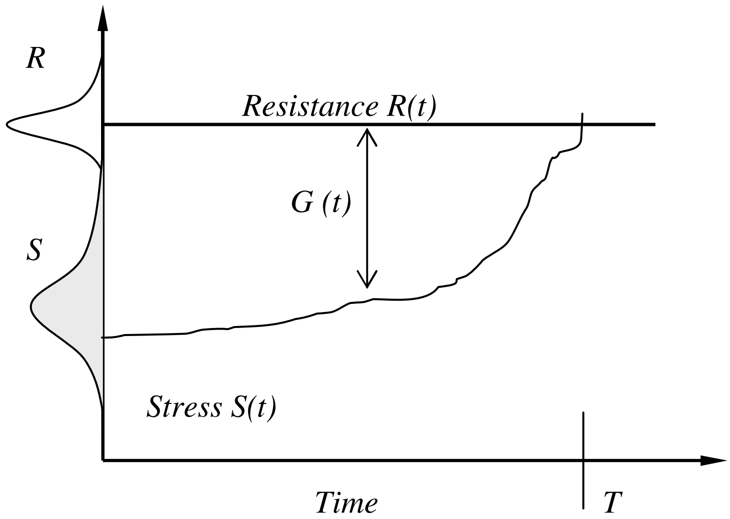


Figure IV. 14 Model of cumulative stochastic damage

It is also possible to use the degradation models mentioned before, with the difference that these models must be rewritten in an incremental way in terms of environmental effects.

$$S(t) = S\big(S_0,\, f(S(\tau), E(\tau), t)\big) \quad \text{with} \qquad 0 \le \tau \le t \tag{IV.13}$$

This expression highlights the coupling between the effect of the environment *S(t)* and the environment itself *E (t),* for all the system history. The safety margin takes the form:

$$G(r,s,t) = R(t) - S(t) = R(t) - S\big(S_0,\, f(S(\tau), E(\tau), t)\big) \tag{IV.14}$$

As an example, this expression can be written for damage *D (t)* the limit value D_L:

$$G(r,s,t) = D_L - D\big(D_0,\, f(D(\tau), E(\tau), t)\big)$$

IV.5 Numerical Example

Gearbox composed of several gears are made of material has the property of ultimate tensile stress 1080 MPa and cov $c_R = 0.05$, the applied stress is normally distributed with mean value equal to 1000 MPa and cov $c_s = 0.2$. Fatigue is the degradation mechanism considered, estimate the life expected at 10% failure for an intrinsically reliable design?

To estimate life (number of load) the following equation is proposed:

$$N = \frac{1}{\int_0^\infty \frac{S(s)}{\xi(s-z)} ds} \tag{IV.15}$$

where *S(s)* is the stress distribution, and $\zeta(s-z)$ is Basquin equation at certain percentage of failure *F%*,

$$\zeta(s-z) = S - N_{F\%} \tag{IV.16}$$

$\zeta(s-z)$ curve relationship determined experimentally, under the following conditions:

- Design must be intrinsically reliable to insure no stress rupture failure can arise.
- The tests results must not demonstrate roller coaster behaviour in terms of hazard [Bom-69], or knee in terms of failure probability (figure IV.15) to insure that there are no initial defects in the test specimens.

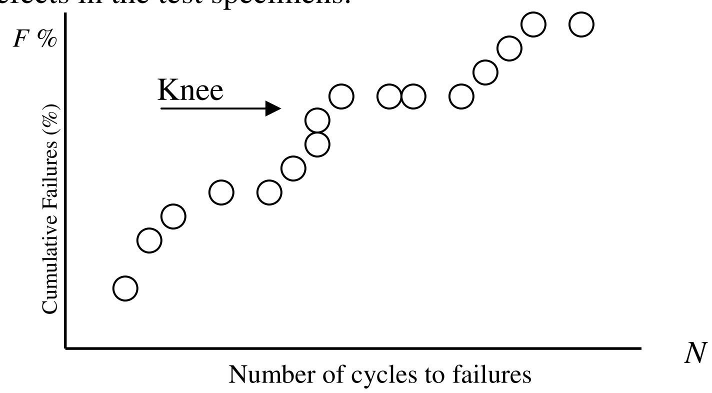


Figure IV. 15 Fatigue tests at constant stress on strips from aero-engine compressor discs [Bom-69].

The life is calculated using an *S-N* curve in conjunction with any damage law is N_F. The *S*-N_F curve corresponding to the required F % failure can be derived. The procedure adopted from [Car-97] is as follows:

1) Derive the *S*-N_F curve for $F_{\%}$ supposed.

PDF is fitted in usual S-N test from 5 tests under constant stresses, to be fitted to Weibull figure IV.16.a and b.

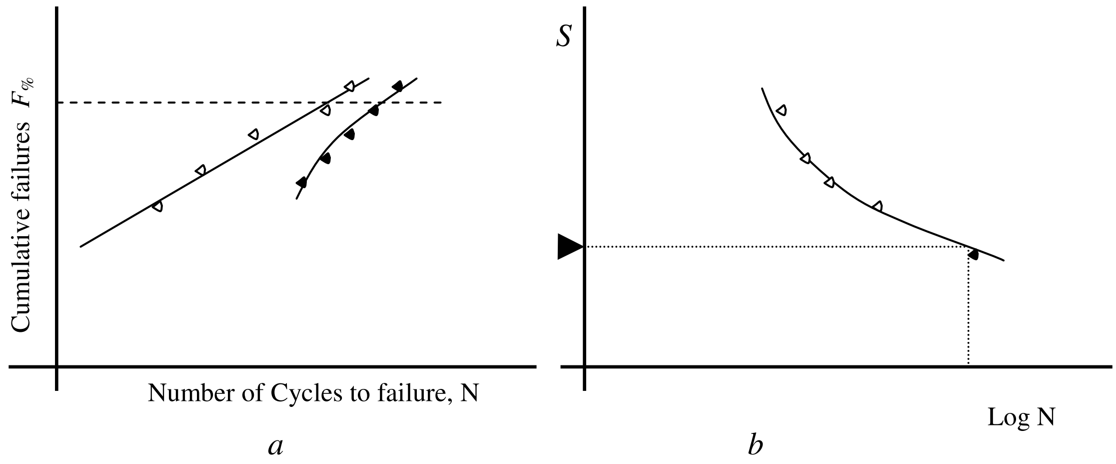


Figure IV. 16 typical test results at constant stress

From which it can be extrapolated to any failure *F%* using maximum likelihood or least square estimation techniques. It is to be noted that variation in the fatigue resistance is equal to the variation in yield or ultimate tensile stress.

Case study introduced of an actual situation will illustrate with a the following data adapted from [Car-97]:

- Reliability will be specified quantitatively by the life of 10 per cent failure, i.e by N_F .
- Gears are made from 832M13 BS,16NCD13 AFNOR.
- Tests under five stress levels based on past experience.
- At each stress level 6 six gears will be tested to failure in bending fatigue. This was far below statistical requirement at any reasonable confidence level.(In view of the cost involved and the length of time required)
- Weibull distribution or other if it is proved more appropriate should be fitted to the life (number of load applications on a tooth before failure) for each stress level and extrapolated to give 10 per cent cumulative failures.
- The five stress levels would then provide five points defining the S-N_F curve, which would be used in design without any factor.
- The gear life recorded, expressed in million of load applications per tooth and given in chronological order of test is as follows.

Stress level	Millions of load application per tooth				
762 Mpa	0.677	10.83	0.533	2.30	0.642
1272 Mpa	0.23	0.279	0.274	0.335	0.392

The results can be examined by Weibull distribution

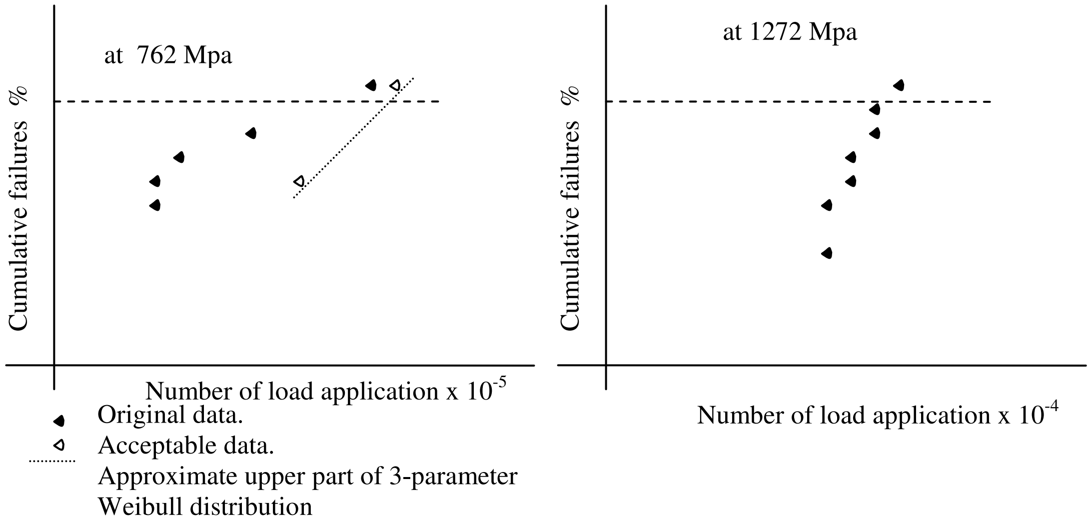


Figure IV. 17 Failure probability under two constant of stress levels

Clearly from the figure IV.17, the data obtained at 1272 MPa stress level was very consistent; no knee like the curve at 762 MPa and it has been accepted. Weibull analysis shows that they can be represented by such a distribution having

- Locating constant $\gamma = 0.2277x10^6$ load applications
- Characteristic life $\eta = 0.2211x10^6$ load applications
- Shaping parameter $\beta_W = 0.506$

Giving a Median value $N_{50} = 0.3349x10^6$ load applications
The correlation of least square best fit Weibull straight line is 0.9748 with 5 degrees of freedom. It is common practice to assume that the median s-N curve for gears is given by

$$N = k.S^m$$

m = -5 according to Merritt relationship.
Substituting the median values at 762 and 1272 MPa into general equation $N = k.S^m$ gives two equations that can be solved for both *m and k*. the values obtained are
$m = -6.3528$ *and* $k = 1.7666x10^{25}$

$$N_{50} = 1.7666 \times 10^{25}.S^{-6.3528}$$

IV.5.1. Estimating $S\text{-}N_F$ curve

Figure IV.18 is common representation of a distributed *S-N* relationship found in the literature, with the distributions of *N* at constant *S* and *S* at constant *N* superimposed.

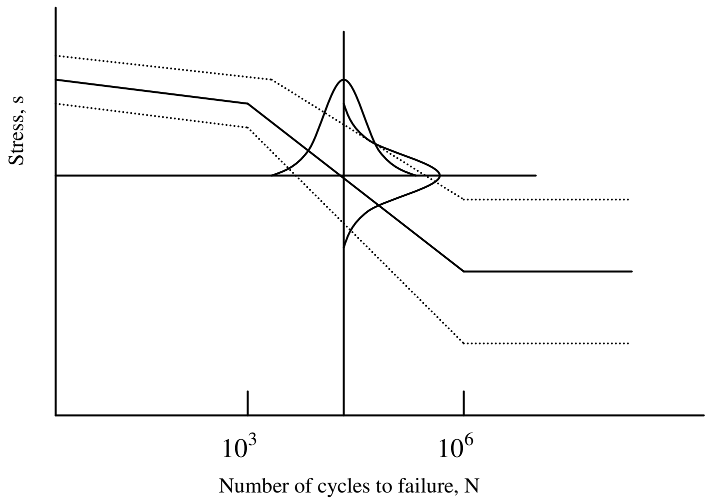


Figure IV. 18 A common representation of a distributed *S-N* relationship

By CDF equation at 1272 MPa for $F=0.1$ we find:

$$F = 1 - e^{-\left(\frac{N - 0.2277x10^6}{0.2211x10^6}\right)^{0.506}}$$

Solving N for $F = 0.1$ gives $N=0.2303\times 10^6$
$(F_{0.1}, N_1) = (0.1, 0.2303x10^6)$ at 1272 MPa as one point of $S\text{-}N_{10}$ curve. $(F_{0.1}, N_2)$ at S_2, $(F_{0.1}, N_3)$ at S_3, $(F_{0.1}, N_4)$ at S_4, $(F_{0.1}, N_5)$ at S_5. From the data available this is the only point can be calculated in this way. When the proposed test plan is complete, this calculation can be repeated at 5 stress levels to obtain the five points required to define the design curve at 0.1 failure value. The equation required is then given by

$$\xi(s - z) = 1.767\times 10^{25}(s + 77)^{-6.353} \tag{IV.17}$$

Then we substitute equation IV.17 in equation IV.15 and the life estimated is:

$$N_{0.10} = 1.023x\ 10^9$$

Hazard then can be evaluated according to the one to one relationship between fatigue resistance and number of load. The results are presented in figure (IV.19).

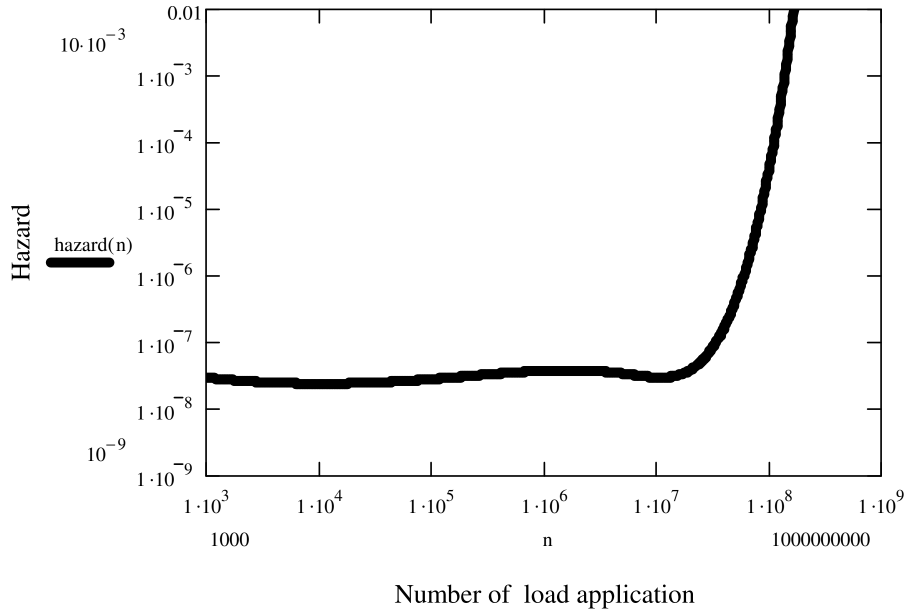


Figure IV. 19 Hazard with load application

Figure IV.19 illustrates how hazard function varies with load application; the sudden death of product is shown. This can be justified that the component will retain its initial strength until rapid crack growth towards the end of life brings about a collapse of resistance over relatively small number of load applications.

IV.6 Conclusion

Modelling degradation (wear out) is mandatory for industrial risks control and life cycle management. It has been shown that the hazard can give interesting measure for the assessment of the degradation level of the mechanical equipment. However, it is required to define a physical degradation model allowing to describe the relationship between the number of load applications, or the elapsed time, and the resistance deterioration of the structure. If this relationship is available, the proposed hazard-based approach can be applied for virtually any mechanical component.

The model can be improved by integrating the epistemic uncertainties in the characterization of the physical degradation models, which may be linked to the testing methodology presented in chapter II.

General Conclusion

In this work, we have developed a methodology for dealing with the life cycle of products, including reliability demonstration tests, hazard-based design in useful life and wear-out considerations.

Concerning the demonstration tests, the question is how to verify the reliability of the product using a small number of tests?

The sample size investigation is carried out according to four approaches based on structural reliability theory: confidence interval, test hypothesis, Bayesian method and compound uncertainties. It has been shown that the compound uncertainty approach has replied to the above question, as it is capable to give the right decision under the assumptions of known resistance distribution and coefficient of variation.

In practice, the most demanding task in the industrial world is how to optimize the cost of product in terms of reliability and quality. Life cycle cost includes several items, where initial and validation costs represent an important part. Solving the cost optimisation equation under the reliability conditions, gives us the optimal number of tests that minimizes the above two costs in the product life cycle. The proposed formulation offers a useful tool that enables the designer and the supplier to find optimal alternatives in production policy.

Concerning the useful period of product life, we seek to answer the question: what is the robust design criterion under repetitive load for time-independent resistance?

We have shown that the hazard-based design offers a robust tool, compared to the failure probability approaches. It is therefore recommended to assign the hazard as a design target for non degraded components ad systems. It gives more realistic results than the extreme value distributions, which lead to largely over-designed components. Although failure probability leads to either under-designed or over-designed products, the hazard can ensure an intrinsically reliable design. Therefore, hazard-based design has got rid of the problem of design sensitivity to the number of load applications. This approach has been generalised to any type of probability density functions, and applied as an optimisation target.

Regarding the wear-out phase, we have the corresponding question: is there a general approach to cope with all degradation mechanisms?

As a matter of fact, there are two approaches to deal with product wear-out: statistical and physical. For mechanical components, the physical approach can give a better understanding of the product degradation mechanisms. On the basis of the stress-resistance model, we have investigated the use of hazard as an indicator of product degradation. In other words, the product failure rate can be monitored and its life can be predicted by the large increase of hazard. The application to fatigue problem shows the applicability of this approach.

This work can be continued by future researches:

- to investigate sequential testing approach;
- to consider the case of random number of load repetitions;
- to extend the optimisation approach to include all the costs involved in the life cycle;
- to generalize the wear-out model to other degradation mechanisms, such as creep, corrosion and wear.

References

[Ahm-08] Ahmed. H., Chateauneuf A. Reliability demonstration for small sample size population, ALT'2008 Second international conference on Accelerated life testing in reliability and quality control.Bordeaux 9-11 juin 2008. pp. 17-22

[Aou-08] Aoues Y., Chateauneuf A. Reliability optimisation of structrural systems by adaptive target safety- Application to RC farme. Journal of Structural safety 30 (2008) 144-161.

[Bec-03] Beck A. Reliability analysis of degrading uncertain structures, with application to fatigue and fracture under random loading. Ph.D. thesis, university of Newcastle, Australia, June 2003.

[Ben-70] Benjamin J.R. and Cornell. Probability, statistics and decision for civil engineers, McGraw Hill, N.Y, (1970).

[Bom-69] Bompas, Smith. The determination of distributions that describe the failures of mechanical components .8th Annals of reliability and maintainability, (1969) 343-356.

[Buc-06] Bucher Ch., Frangopol D.M. Optimization of lifetime maintenance strategies for deteriorating structures considering probabilities of violating safety, condition, and cost thresholds. Probabilistic Engineering Mechanics 21 (2006) 1–8.

[Car-86] Carter A.D.S. Mechanical reliability, Macmillan Education LTD (1986).

[Car-97] Carter A.D.S.. Mechanical reliability and design, Macmillan Press LTD (1997).

[Cia-98] Ciampoli M. Reliability assessment of structural systems subjecte to deterioration over time. Structural Safety and Reliability, (1998) 81-88.

[Cha-06] Chateauneuf A. Optimisation de la conception et dimensionnement d'un produit à l'aide d'une extension de l'approche résistance/contrainte. IMdR Projet n° P05-1, 2006.

[Cha-08] Chateauneuf A., Grzeskowiak H., Pierrat L. Modélisation des dégradations des structures et optimisation de leurs inspections IMdR Projet n° P07-1.2008.

[Clo-34] Clopper .CJ, Pearson. ES. The use of confidence or fiducially limits, illustrated in the case of binomial. Biometrika 26: (1934) 404-413.

[Con-01] Lloyed W.C. Reliability improvement with design of tests, Marcel Dekker, INC,(2001).

[Dhi-88] Dhillon. B.S. Mechanical Reliability: Theory Models and application, Aiaa Education Series (1988).

[Dow-92] Dowlatshahi, S. Product design in concurrent engineering environments an optimisation approach, journal of production research,30/8:1803-1818, 1992.

[Du-02] Xiaonping Du. X, Chen. W.Sequential Optimization and Relaibility Assessment method for Efficient Probabilistic Design. ASME Design Automation Conference, Montreal, Canada.

[Eng-95] English J.R, Yan Li, Landers.T.L. A modified bathtub curve with latent failures, Proceeding Annual Reliability and maintainability symposium (1995).

[Gue-99] Guedes-Soares C., Garbatov Y. Reliability of maintained, corrosion protected plates subjected to non-linear corrosion and compressive loads. Marine Structures 12 (1999) 425-445.

[Hag-91] Hagen O., Tvedt L. vector process out-crossing as parallel system sensitivity measure. Journal of Engineering Mechanics, 117(10) (1991) 2201-2220.

[Ham-88] Hamberg O., Tosney W. The effectiveness of satellite environmental acceptance tests. Proceedings of the Aerospace testing seminar, October (1988).

[Has-74] Hasofer, Lind. Exact and Invariant Second Moment Code Format "journal of Engineering Mechanics Division, ASCE, Vol. 100, (1974), 111-121.

[Hob-00] Hobbs, G. K, Accelerated Reliability Engineering: HALT and HASS, Wiley, Chich- ester, West Sussex, England (2000).

[Jen-82] Jensen F., Peterson N.E .Burn - in : an engineering approach to the design and analysis of burn-in procedures, Wiley(1982).

[Kap-77] Kapur K.C., Lamberson L.R. Reliability in Engineering Design, Wiley, Hoboken, NJ, (1977).

[Kec-99] Kececioglu, Sun. Environmental stress screening, its qualification, optimization and management. Ed Prentice Hall, (1999).

[Kha-02] Kharmanda G., Mohamed A., Lemaire M. Efficient reliability based design optimization using a hybrid sapce with application to finite element analysis. Structural Multidisciplinary Optimization 24, 233-245Springer-Verlag.

[Kle-04] Kleyner A., Sandborn P.,Boyle J. Minimisation of life cycle costs through optimisation of the validation program- A test sample size and warranty cost approach. Proceedings of annual reliability and maintainability symposium, Los Angeles, CA,PP.553-557, 2004.

[Kle-07] Kleyner A., Sandborn P. Minimising life cycle cost by managing product reliability via validation plan and warranty return cost. International journal of production economics, 2007.

[Kuo-98] Kuo W., Chien W. T., Kim T. Reliability, Yield, and Stress Burn-in, Kluwer Academic, Norwell, MA (1998).

[Lan-05] André Lannoy. Henri Procaccia. Evaluation et maîtrise du vieillissement industriel, Lavoisier-2005.

[Lew-94] Lewis E.E.. Introduction to reliability engineering 2 edition, John Wiley& Sons, Inc, (1994).

[Mad-86] Madsen H.O., Krenk S. and Lind N.C. Methods of structural safety. Englewood Cliffs, Prentice-Hall; (1986).

[Mad-86] Methods of structural safety, Prentice-Hall, Inc., Englewood Cliffs, NJ 07632.

[Mee-98] Meeker W.Q. and Escobar L.A.Statistical Methods for Reliability Data. John Wiley & Sons, (1998).

[Mel-99] Melchers R.E.. Structural reliability analysis and prediction 2nd Edition,John Wiley & Sons Ltd, 1999.

[Mau-96] Lemaire M., Muzeau J.P., Chateauneuf A., Mitteau J.C., Fiabilité et sûreté de fonctionnement des systèmes Mécaniques, Cours au Collège de Polytechnique, (1996).

[Mor-01] Mori Y., Nonaka M. LRFD for assessment of deteriorating existing structures. Structural Safety 23 (2001) 297–313.

[Mos-77] Moses F. Strucutral System reliability and optimization. Computer Strucutre 7, 283-290, 1977.

[Nak-02] Nakanishi S., Nakavasu H. Reliability design of structural system with cost effectiveness during life cycle. Computers and Industrial Engineering 42(2002) 447-456.

[Nel-90] Nelson W. Accelerated Testing: Statistical models, test plans, and data analysis. John Wiley and sons, New York,(1990).

[Pla-99] Place C.S., Strutt J.E., Allsopp, Irving P.E., Trille C.. Quality and reliability Engineering international.15:69-78(1999)

[Pon-05] Poncet A., Schimmerling P., Lyonnet P., Jouzeau C and C.Tavasson. Le dimensionnement des essais de validation par la méthode “contrainte résistance”. Congrès Lambda-Mu 12, Octobre (2005) 12-14.

[Rol-94] Roland. W.S., Dietrich D.L., 1994. Proceedings annul Reliability and maintainability symposium.

[Sch-91] Schall G., Faber M. Rackwitz R. the ergodicity assumption for sea states in the reliability assessment of offshore structures. J. Offshore Mechanics and Arctic Engineering, 113(3) (1991) 241-246.

[Shi-83] Shinozuka. M. Basic analysis of structural safety , Journal of structural Engineering, ASCE, Vol.109,No 3, Mars, PP. 721-740, (1983).

[Smi-94] Roland W.Smith. Duane L. Dietrich.The bathtub curve: An alternative explanation. Proceeding annual reliability and maintainability symposium 1994.

[Was-02] Wasserman G. Reliability verification, testing, and analysis in Engineering design, Marcel Dekker, INC, New York. Basel (2002).

[Won-90] Wong K.L. IEEE Transaction Reliability “Demonstrating reliability and reliability growth with environmental stress screening data”, Proceeding Annual reliability and maintainability symposium, (1990).

[Yan-07] Yang G. Life cycle reliability engineering, John Wiley & Sons, 2007.

RESUME

Le processus de dimensionnement des structures et des systèmes mécaniques comportent de plusieurs étapes, allant de la définition des conditions et des besoins tout au long du cycle de vie, en vue de la spécification de la capacité et de la résistance requises pour accomplir les missions escomptées. La fiabilité figure parmi les objectifs les plus importants pour les fabricants, en plus de l'aspect économique, facteur clé, qui influence largement le processus de conception. Dans ce contexte, la conception doit être élaborée afin de définir le meilleur compromis entre la fiabilité et le coût. Ce qui implique une étude précise et détaillée de tout le cycle de vie du produit, de la naissance jusqu'à la mise au rebus.

Cette étude couvre les différentes phases du cycle de vie du produit, en intégrant la nécessité de démontrer la fiabilité du produit avant de commencer la production en série, sous des contraintes de coût et de délais.

Ce travail vise à donner des éléments de réponse aux trois questions suivantes :

- Comment peut-on démontrer la fiabilité du produit à partir de quelques essais ? Parmi les quatre approches considérées, la méthode de composition des incertitudes montre sa robustesse pour démontrer la fiabilité du produit, sans pour autant conduire à un surdimensionnement excessif.

- Quel est le critère permettant une conception robuste sous des charges répétitives pour un système non dégradable ? Dans la phase utile du cycle de vie du produit, la défaillance est principalement due à la variabilité des charges appliquées lorsque la résistance n'est pas dégradée. Le modèle d'interférence contrainte-résistance considère la probabilité de défaillance comme cible de conception. Cependant, pour le cas des charges répétitives, ce critère est sensible au nombre d'applications de ces charges. Pour cela, la conception basée sur le hasard est proposée comme outil robuste pour la conception des composants intrinsèquement fiables.

- Quelle est l'approche générale permettant de traiter les mécanismes de dégradation ? Dans la phase de vieillissement, la modélisation de la dégradation est obligatoire pour plusieurs raisons, telles que la maîtrise des risques industriels et la gestion du cycle de vie. La fonction de hasard fournit un indicateur approprié pour la prévision de l'état de dégradation et par conséquent, l'estimation de la durée de vie résiduelle.

ABSTRACT

The process of designing and producing mechanical and structural systems consists of several stages, starting from defining the requirements and the demands throughout the life cycle, that must be supported to determine the capacity or resistance needed to fulfil the equipment mission. The reliability is the one of the most important goals that manufacturers seek, while the economical aspect is a key factor and it has a great deal influence on this process. Therefore, the best design has to be carried out, in order to achieve the paradox of reliable products with minimal costs. This implies careful and exact investigation along the product life-cycle, from birth to death.

This study encompasses the different phases of product life cycle, starting from the necessity to demonstrate the product reliability before starting the mass production under the constraints of economy and time.

This work aims to answer the following three questions:

- How can we demonstrate the product reliability on the basis of few tests? Among the four approaches considered in this study, the method of compound uncertainties shows its robustness to demonstrate the product reliability, without implying unnecessary over-design.

- What is the robust design criterion under repetitive load for time-independent resistance? In the useful phase of product life cycle, failure is assigned to load variability under the assumption of non-degraded resistance. Stress-resistance model considers failure probability as a design target to be achieved; however, for the case of repetitive loading, this criterion is sensitive to the number of load applications. The present works shows that hazard is almost constant and gives a robust design criterion.

- Is there a general approach that can cope with all degradation mechanisms? In the wear out phase, modelling degradation is mandatory for several reasons such as industrial risks control and life cycle management. The hazard function gives an appropriate indicator for the prediction of the degradation state and consequently, the estimation of residual product life.